Exercise 1: Metric Learning

a) Given a valid metric $D_M$, is $D_M^2$ also a metric? Why?

b) Given a matrix $X$ of $n$ data points $x_i \in \mathbb{R}^d$, show how computing the eigen-decomposition of the covariance of $X$ is equivalent to computing the singular value decomposition of $X$.

c) What is the difference between metric learning and kernel learning? When would you prefer to use a kernel method over a metric learning method?

d) In Neighborhood Component Analysis, we define a stochastic neighbor selection rule. The probability that a data point $j$ is selected as neighbor of point $i$ is given by:

$$p_{ij} = \frac{\exp\{-||Lx_i - Lx_j||^2\}}{\sum_{k\neq i} \exp\{-||Lx_i - Lx_k||^2\}}$$

namely a softmax over the squared distances to all points in the transformed space. The goal is to maximize

$$f(L) = \sum_i \sum_{j \in C_i} p_{ij}$$

namely the probability that the neighbors that will be selected for each point $i$ will belong to the same class $C_i$. Can you derive the gradient of $f(L)$?

e) What is the difference between LDA and NCA?

f) The KL-divergence measures the (dis-)similarity of two probability distributions. It is defined as:

$$D_{KL}(p||q) = \int p(x) \log \left(\frac{p(x)}{q(x)}\right) dx$$

Is the KL-divergence a metric? Why?