Convex Optimization for Machine Learning and Computer Vision

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Winter Semester 2018/19

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Weekly Exercises 9

Room: 01.09.014

Wednesday, 09.01.2019, 12:15-14:00

Submission deadline: Monday, 07.01.2019, 16:15, Room 01.09.014

Theory: PDHG and ADMM (Due: 07.01.2019) (8+4 Points)

Exercise 1 (6 Points)**.** Let $S \in \mathbb{R}^{m\times m}, T \in \mathbb{R}^{n\times n}$ be 2 symmetric positive definite matrices and $K \in \mathbb{R}^{n\times m}$, show that

$$M = \begin{bmatrix} S & -K^\top \\ -K & T \end{bmatrix} \text{ is symmetric positive definite } \Leftrightarrow S - K^\top T^{-1} K > 0 \tag{1}$$

Hint: Consider the Schur complement of M and prove that a block diagonal matrix is symmetric positive definite if and only if all of its diagonal blocks are symmetric positive definite.

Exercise 2 (6 Points)**.** (ADMM update derivation for Robust PCA): we consider the following optimization problem (the programming exercise below gives the context):

$$\operatorname*{argmin}_{\substack{A\in\mathbb{R}^{n_1\times n_2}\\ B\in\mathbb{R}^{n_1\times n_2}\\ M\in\mathbb{R}^{n_1\times n_2}}} \|A\|_{\text{nuc}} + \lambda \|B\|_1 + \delta\{\|M-Z\|_{\text{fro}} \leq \epsilon\} + \delta\{A+B-M=0\} \tag{2}$$

where $Z \in \mathbb{R}^{n_1\times n_2}$ is a given matrix, $\|\|_{\text{fro}}$ is the Frobenius norm, $\|\|_{\text{nuc}}$ is the nuclear norm and $\delta\{\}$ is the indicator function.

The M here can be considered as a replacement variable and we introduce the Lagrangian multiplier Y to construct the augmented Lagrangian:

$$\mathcal{L}(A,B,M,Y) = \|A\|_{\text{nuc}} + \lambda\|B\|_1 + \delta\{\|M-Z\|_{\text{fro}} \leq \epsilon\} + \langle Y, A+B-M\rangle + \frac{\rho}{2}\|A+B-M\|_{\text{fro}}^2 \tag{3}$$

You are asked to write down the ADMM updates to solve above augmented Lagrangian function on A, B, M, Y.

Hint: This is a more general form than what we see in the lecture. Nevertheless, you can write down the iterative updates for A, B, M, Y sequentially similar to the one in the lecture.

Programming: Robust Principal Component Analysis(Due date: 07.01.2019) (12 Points)

Exercise 3 (12 Points). Given several frames from a video, your task is to separate the foreground and background by solving an optimization problem: Assume that each frame is an image with $m \times n$ pixels and this video has n_2 number of frames. By vectorizing each frame, we can create a matrix $Z \in \mathbb{R}^{n_1 \times n_2}$, where $n_1 = m \times n$.

Inspired by the idea of PCA, we want to decompose the original matrix Z into two matrices A and B with the same dimension. The matrix A should contain the information of background pixels while B should contain the information of foreground ones. We hope that $A+B$ will recover the original video Z, i.e. $A+B = Z$. However, considering the noise in Z, an intermediate matrix $M := A+B$ is introduced. Instead of recovering the exact Z, we relax the constrain by requiring $\|M - Z\|_{\text{fro}} \leq \epsilon$, where ϵ is a predefined variable controlling the trade-off between the fidelity of the decomposition and the robustness to the noise.

Therefore, we could construct the following optimization problem:

$$\operatorname{argmin}_{\substack{A \in \mathbb{R}^{n_1 \times n_2} \\ B \in \mathbb{R}^{n_1 \times n_2} \\ M \in \mathbb{R}^{n_1 \times n_2}}} \|A\|_{\text{nuc}} + \lambda \|B\|_1 + \delta\{\|M - Z\|_{\text{fro}} \leq \epsilon\} + \delta\{A + B - M = 0\} \quad (4)$$

where $\|\|_{\text{fro}}$ is the Frobenius norm, $\|\|_{\text{nuc}}$ is the nuclear norm and $\delta\{\}$ is the indicator function.

Since A contains background of each frame and the background keeps the same, A should be a low-rank matrix. Therefore, the nuclear norm is used to constraint A to be a low-rank matrix. The l_1 norm of B requires B to be sparse.

You are asked to apply ADMM to solve this energy function. The M here can be considered as a replacement variable and we introduce the Lagrangian multiplier Y to construct the augmented Lagrangian:

$$\mathcal{L}(A, B, M, Y) = \|A\|_{\text{nuc}} + \lambda \|B\|_1 + \delta\{\|M - Z\|_{\text{fro}} \leq \epsilon\} + \langle Y, A + B - M \rangle + \frac{\rho}{2} \|A + B - M\|_{\text{fro}}^2 \quad (5)$$

Then use ADMM to solve:

$$\operatorname{argmin}_{A,B,M,Y} \mathcal{L}(A, B, M, Y) \quad (6)$$