Convex Optimization for Machine Learning and Computer Vision

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Weekly Exercises 3

Room: 02.09.023
Friday, 17.11.2017, 09:15-11:00
Submission deadline: Monday, 13.11.2017, 10:15, Room 02.09.023

Theory: Lipschitz continuity, fixed point iterations and gradient descent (12+4 Points)

Exercise 1 (4 Points)**.** We call a function $E : \mathbb{R}^n \to \mathbb{R}$ *absolutely one-homogeneous* if

$$E(\alpha u) = |\alpha| E(u)$$

holds for all $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Prove that

$$\partial E(u) = \{p \in \mathbb{R}^n \mid \langle p, u\rangle = E(u), \quad E(v) \geq \langle p, v\rangle \ \forall v \in \mathbb{R}^n\}.$$

Exercise 2 (4 Points)**.** Find examples for the following functions and explain why your example is correct:

- A continuously differentiable convex function that is not L-smooth.
- A Lipschitz continuous function that is not a contraction.
- A function that is not differentiable, but Lipschitz continuous.
- A convex L-smooth function E and a step size τ for which G defined by $G(u) = u - \tau \nabla E$ is not a non-expansive function.

Exercise 3 (4 Points)**.** Show that for any $a, b \in \mathbb{R}^n$, $\theta \in \mathbb{R}$ it holds that

$$\|(1-\theta)a + \theta b\|^2 = (1-\theta)\|a\|^2 + \theta\|b\|^2 - \theta(1-\theta)\|a-b\|^2$$

Exercise 4 (4 points)**.** Let the function $E : \mathbb{R}^n \to \mathbb{R}$ be given as

$$E(u) := t(u) + h(u).$$

where the function $h : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$h(u) := g(Du), \qquad g(v) = \sum_{i=1}^{2n} \varphi(v_i), \qquad \varphi(x) = \sqrt{x^2 + \epsilon^2},$$

with $D \in \mathbb{R}^{2n \times n}$ being a finite difference gradient operator and $t : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$t(u) := \frac{\lambda}{2}\|u - f\|^2.$$

1. Show that the function E is L-smooth with $L = \lambda + \frac{\|D\|^2}{\epsilon}$.
2. Show that the function E is m-strongly convex, with $m = \lambda$.

Programming: Image denoising (12 Points)

Exercise 5 (12 Points). Denoise the noisy input image f, given in the file `noisy_input.png` by minimizing the energy from Ex. 3:

$$E(u) = \frac{\lambda}{2} \|u - f\|^2 + \sum_{i=1}^{2n} \sqrt{(Du)_i^2 + \epsilon^2}$$

with gradient descent. To guarantee convergence choose your step size τ so that

$$0 < \tau \leq \frac{2}{m + L}.$$

Use MATLABs `normest` to estimate the norm $\|D\|$ of your finite difference gradient operator D. Here, n is the number of pixels times the number of color channels.