

Variational Methods for Computer Vision: Solution Sheet 4

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Part I: Theory

1. In the one-dimensional case we write $\Omega = [a, b] \subset \mathbb{R}$, meaning that the boundary $\partial\Omega = \{a, b\}$. We proceed accordingly to the lecture:

$$\begin{aligned}
 \left. \frac{dE(u)}{du} \right|_h &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (E(u + \varepsilon h) - E(u)) \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_a^b \mathcal{L}(u + \varepsilon h, u' + \varepsilon h', u'' + \varepsilon h'') - \mathcal{L}(u, u', u'') dx \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_a^b \mathcal{L}(u, u', u'') + \frac{\partial \mathcal{L}}{\partial u} \varepsilon h + \frac{\partial \mathcal{L}}{\partial u'} \varepsilon h' + \frac{\partial \mathcal{L}}{\partial u''} \varepsilon h'' + \mathcal{O}(\varepsilon^2) - \mathcal{L}(u, u', u'') dx \\
 &= \int_a^b \frac{\partial \mathcal{L}}{\partial u} h + \frac{\partial \mathcal{L}}{\partial u'} h' + \frac{\partial \mathcal{L}}{\partial u''} h'' dx \\
 &= \int_a^b \frac{\partial \mathcal{L}}{\partial u} h dx - \int_a^b h \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} dx - \int_a^b \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u''} h' dx + \left[h \frac{\partial \mathcal{L}}{\partial u'} \right]_a^b + \left[h' \frac{\partial \mathcal{L}}{\partial u''} \right]_a^b \\
 &= \int_a^b h \left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} + \frac{d^2}{dx^2} \frac{\partial \mathcal{L}}{\partial u''} \right) dx - \left[h \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u''} \right]_a^b + \left[h \frac{\partial \mathcal{L}}{\partial u'} \right]_a^b + \left[h' \frac{\partial \mathcal{L}}{\partial u''} \right]_a^b.
 \end{aligned}$$

2. (a)

$$\begin{aligned}
 \operatorname{div}(fg) &= \sum_{i=1}^n \frac{\partial (fg_i)}{\partial x_i} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} g_i + f \frac{\partial g_i}{\partial x_i} \right) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} g_i + f \sum_{i=1}^n \frac{\partial g_i}{\partial x_i} \\
 &= \nabla f \cdot g + f \operatorname{div} g.
 \end{aligned}$$

- (b) By the divergence theorem, we know that $\int_{\Omega} \operatorname{div}(fg) dx = \int_{\partial\Omega} \langle fg, n \rangle ds$. Replacing the divergence with the result from question 2a yields

$$\int_{\Omega} \langle \nabla f, g \rangle dx + \int_{\Omega} f \operatorname{div} g dx = \int_{\partial\Omega} f \langle g, n \rangle ds,$$

which is easily transformed into the desired result,

$$\int_{\Omega} \langle \nabla f, g \rangle dx = \int_{\partial\Omega} f \langle g, n \rangle ds - \int_{\Omega} f \operatorname{div} g dx.$$

3. To simplify notation, we write $(x \ y \ z)^{\top} = p \in \mathbb{R}^3$, $(u_x \ u_y \ u_z)^{\top} = \nabla u$ and

$$\frac{\partial \mathcal{L}}{\partial \nabla u} = \left(\frac{\partial \mathcal{L}}{\partial u_x} \quad \frac{\partial \mathcal{L}}{\partial u_y} \quad \frac{\partial \mathcal{L}}{\partial u_z} \right)^{\top}.$$

$$\begin{aligned}
\left. \frac{dE(u)}{du} \right|_h &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (E(u + \varepsilon h) - E(u)) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} \mathcal{L}(u + \varepsilon h, \nabla(u + \varepsilon h)) - \mathcal{L}(u, \nabla u) \, dp \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} \mathcal{L}(u, \nabla u) + \frac{\partial \mathcal{L}}{\partial u} \varepsilon h + \frac{\partial \mathcal{L}}{\partial u_x} \varepsilon \frac{\partial h}{\partial x} + \frac{\partial \mathcal{L}}{\partial u_y} \varepsilon \frac{\partial h}{\partial y} + \frac{\partial \mathcal{L}}{\partial u_z} \varepsilon \frac{\partial h}{\partial z} + \mathcal{O}(\varepsilon^2) - \mathcal{L}(u, \nabla u) \, dp \\
&= \int_{\Omega} \frac{\partial \mathcal{L}}{\partial u} h \, dp + \int_{\Omega} \left\langle \nabla h, \left(\frac{\partial \mathcal{L}}{\partial u_x} \quad \frac{\partial \mathcal{L}}{\partial u_y} \quad \frac{\partial \mathcal{L}}{\partial u_z} \right)^T \right\rangle \, dp \\
&= \int_{\Omega} \frac{\partial \mathcal{L}}{\partial u} h \, dp - \int_{\Omega} h \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial u_x} \quad \frac{\partial \mathcal{L}}{\partial u_y} \quad \frac{\partial \mathcal{L}}{\partial u_z} \right)^T \, dp \stackrel{\text{ff}}{=} + \int_{\partial \Omega} h \left\langle \left(\frac{\partial \mathcal{L}}{\partial u_x} \quad \frac{\partial \mathcal{L}}{\partial u_y} \quad \frac{\partial \mathcal{L}}{\partial u_z} \right)^T, n \right\rangle \, ds \\
&= \int_{\Omega} h \left(\frac{\partial \mathcal{L}}{\partial u} - \left(\frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial u_x} + \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{\partial u_y} + \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{\partial u_z} \right) \right) \, dp + \int_{\partial \Omega} h \left(\frac{\partial \mathcal{L}}{\partial u_x} n_x + \frac{\partial \mathcal{L}}{\partial u_y} n_y + \frac{\partial \mathcal{L}}{\partial u_z} n_z \right) \, ds.
\end{aligned}$$

Hence we can write the Euler-Lagrange equations as

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial u} - \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial \nabla u} \right) &= 0, & \text{on } \Omega, \\
\left\langle \frac{\partial \mathcal{L}}{\partial \nabla u}, n \right\rangle &= 0, & \text{on } \partial \Omega,
\end{aligned}$$

where n denotes the normal vector on the boundary $\partial \Omega$.

Note that if the boundary term vanishes, we have

$$\langle E(u), h \rangle = \left. \frac{dE(u)}{du} \right|_h = \left\langle \frac{\partial \mathcal{L}}{\partial u} - \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial \nabla u} \right), h \right\rangle, \quad \forall h,$$

from which follows that

$$E(u) = \frac{\partial \mathcal{L}}{\partial u} - \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial \nabla u} \right).$$

Remark: To see the last equality, consider for scalar products in general

$$\begin{aligned}
\langle f, h \rangle &= \langle g, h \rangle, \forall h \\
\Leftrightarrow \langle f - g, h \rangle &= 0, \forall h \\
\Rightarrow \langle f - g, f - g \rangle &= 0, \\
\Rightarrow f &= g.
\end{aligned}$$

4. (a) Using the same notation as previously, we have that $\mathcal{L}(u, \nabla u) = \sqrt{u_x^2 + u_y^2}$ and

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial u_x} &= \frac{u_x}{\sqrt{u_x^2 + u_y^2}}, \\
\frac{\partial \mathcal{L}}{\partial u_y} &= \frac{u_y}{\sqrt{u_x^2 + u_y^2}},
\end{aligned}$$

with which we evaluate

$$\frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial \nabla u} \right) = -\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right).$$

Thus the Euler-Lagrange equations are given as the following:

$$\begin{aligned}\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) &= 0, & \text{on } \Omega, \\ \left\langle \frac{\nabla u}{|\nabla u|}, n \right\rangle &= 0, & \text{on } \partial\Omega.\end{aligned}$$

- (b) Similarly to the previous exercise, by considering partial derivatives by u_x and u_y and expanding the matrix products, we derive for

$$\mathcal{L}(u, \nabla u) = \sqrt{(\nabla u)^\top D \nabla u} = \sqrt{(u_x \ u_y) \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}}$$

the partial derivative

$$\frac{\partial \mathcal{L}}{\partial \nabla u} = \left(\frac{\partial \mathcal{L}}{\partial u_x} \ \frac{\partial \mathcal{L}}{\partial u_y} \right)^\top = \dots = \frac{(D + D^\top) \nabla u}{2\sqrt{(\nabla u)^\top D \nabla u}},$$

and thus

$$\frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \operatorname{div} \left(\frac{\partial \mathcal{L}}{\partial \nabla u} \right) = - \operatorname{div} \left(\frac{(D + D^\top) \nabla u}{2\sqrt{(\nabla u)^\top D \nabla u}} \right),$$

with which we arrive at the following Euler-Lagrange equations:

$$\begin{aligned}\operatorname{div} \left(\frac{(D + D^\top) \nabla u}{2\sqrt{(\nabla u)^\top D \nabla u}} \right) &= 0, & \text{on } \Omega, \\ \left\langle \frac{(D + D^\top) \nabla u}{2\sqrt{(\nabla u)^\top D \nabla u}}, n \right\rangle &= 0, & \text{on } \partial\Omega.\end{aligned}$$