# Variational Methods for Computer Vision: Exercise Sheet 2

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## Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. A multidimensional filter ist called separable, if it can be decomposed in one dimensional filter operations. Prove that the convolution of an image $f$ with a Gaussian kernel $K$ of standard deviation $\sigma > 0$,
$$K(x, y) := \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right),$$
can be written as the convolution with two one-dimensional filters:
$$k_1(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad \text{and} \quad k_2(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right).$$
Hence:
$$(f * K)(x, y) = ((f * k_1) * k_2)(x, y),$$
Explain why the separability of a filter is a desirable property.

2. Let $f \in C^2(\Omega; \mathbb{R})$ with $\Omega \subset \mathbb{R}^2$ be a real valued function and let $R \in \mathrm{SO}(2)$ be a rotation matrix. Prove that the magnitude of the gradient and the Laplace operator are rotationally covariant by showing the following identities:

   (a) $\nabla(f \circ R) = R^\top \circ (\nabla f) \circ R$

   (b) $\|\nabla(f \circ R)\| = \|(\nabla f) \circ R\|$

   (c) $\Delta(f \circ R) = (\Delta f) \circ R$

   *Reminders:*

   - $R \in \mathrm{SO}(2)$ denotes $2 \times 2$ matrices with $\det(R) = 1$ and $R^\mathsf{T} R = R R^\mathsf{T} = I$ and can be written as
   $$R = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$
   for some $\alpha \in [0, 2\pi)$.
   - The multivariate chain-rule for the Jacobian is
   $$J_{f \circ g}(a) = J_f(g(a)) \circ J_g(a).$$
   - For matrix multiplication and the dot product we have the following identity
   $$\langle Ax, y \rangle = (Ax)^\top y = x^\top A^\top y = \langle x, A^\top y \rangle.$$

3. The general diffusion equation can be written as follows

$$\begin{aligned} \partial_t u &= \operatorname{div}(g \cdot \nabla u), && \text{in } \Omega \times [0, \infty), \\ \partial_\nu u &= 0, && \text{on } \partial\Omega \times [0, \infty) \\ u(x, 0) &= u_0(x), && \text{for } x \in \Omega, \end{aligned}$$

where $u \in C^2(\Omega \times \mathbb{R}_0^+; \mathbb{R})$ with $\Omega \subset \mathbb{R}^2$ describes the complete diffusion process and solves the partial differential equation. Note that $\partial_\nu u = \langle \nabla u, \nu \rangle$ is the gradient in normal direction $\nu$. Prove the following identities:

(a) *linear homogeneous diffusion*:

$$\operatorname{div}(g \cdot \nabla u) = g \Delta u, \qquad g \in \mathbb{R}.$$

(b) *linear inhomogeneous diffusion*:

$$\operatorname{div}(g \cdot \nabla u)(x) = g(x)\Delta u(x) + \langle \nabla g(x), \nabla u(x) \rangle, \qquad g \in C^1(\Omega; \mathbb{R}).$$

## Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. Download the archive file `vmcv_ex02.zip` from the homepage and unzip it in your home folder. Use the template file `diffusion_filter.m` to implement a nonlinear diffusion filter and complete the missing code in ll. 60. Test the script on the image `lena.png`. You can use explicit time discretization and the space discretization presented in the lecture.

2. Experiment with the different diffusivity variants and step sizes. What do you observe?

3. Create a video using the provided script `vmcv_ex02.m` and compare the results.