Variational Methods for Computer Vision: Solution Sheet 7

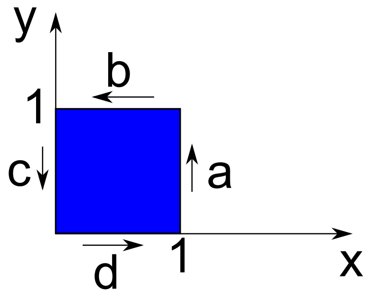
Exercise: December 12, 2017

Part I: Theory

1. (a) The line integral of a vector field V along a curve $\gamma(t)$ is defined as

$$\int_\gamma V(s)\mathrm{d}\vec{s} = \int_0^T \langle V(\gamma(t)), \dot{\gamma}(t)\rangle \,\mathrm{d}t\,,$$

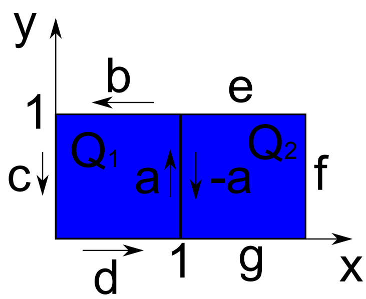
so we have to integrate over the scalar product of V with the tangent vector to the curve at each point of the curve. For a square, the tangent vectors are $(0, \pm 1)$ and $(\pm 1, 0)$.



We start by evaluating the left hand side of the equation:

$$\begin{aligned}
\int_Q \operatorname{curl} V \mathrm{d}x\mathrm{d}y &= \int\limits_Q v_x(x,y) - u_y(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int\limits_0^1\int\limits_0^1 v_x(x,y)\mathrm{d}x\mathrm{d}y - \int\limits_0^1\int\limits_0^1 u_y(x,y)\mathrm{d}y\mathrm{d}x \\
&= \int\limits_0^1 v(x,y)|_{x=0}^{x=1}\,\mathrm{d}y - \int\limits_0^1 u(x,y)|_{y=0}^{y=1}\,\mathrm{d}x \\
&= \int\limits_0^1 v(1,y)\mathrm{d}y - \int\limits_0^1 v(0,y)\mathrm{d}y - \int\limits_0^1 u(x,1)\mathrm{d}x + \int\limits_0^1 u(x,0)\mathrm{d}x \\
&= \underbrace{\int\limits_0^1 v(1,y)\mathrm{d}y}_{\int_a V(s)d\vec{s}} + \underbrace{\int\limits_1^0 v(0,y)\mathrm{d}y}_{\int_c V(s)d\vec{s}} + \underbrace{\int\limits_1^0 u(x,1)\mathrm{d}x}_{\int_b V(s)d\vec{s}} + \underbrace{\int\limits_0^1 u(x,0)\mathrm{d}x}_{\int_d V(s)d\vec{s}} \\
&= \oint\limits_{\partial Q} V(s)\mathrm{d}\vec{s}.
\end{aligned}$$

(b) To show the principle, we first join two squared of same side length that touch in one side:



$$
\begin{aligned}
&\int_{Q_1} v_x(x,y) - u_y(x,y)\mathrm{d}x\mathrm{d}y + \int_{Q_2} v_x(x,y) - u_y(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_a V(s)\mathrm{d}\vec{s} + \int_b V(s)\mathrm{d}\vec{s} + \int_c V(s)\mathrm{d}\vec{s} + \int_d V(s)\mathrm{d}\vec{s} \\
&- \int_a V(s)\mathrm{d}\vec{s} + \int_g V(s)\mathrm{d}\vec{s} + \int_f V(s)\mathrm{d}\vec{s} + \int_e V(s)\mathrm{d}\vec{s} \\
&= \int_b V(s)\mathrm{d}\vec{s} + \int_c V(s)\mathrm{d}\vec{s} + \int_d V(s)\mathrm{d}\vec{s} + \int_g V(s)\mathrm{d}\vec{s} + \int_f V(s)\mathrm{d}\vec{s} + \int_e V(s)\mathrm{d}\vec{s} \\
&= \oint_{\partial(Q_1 \cup Q_2)} V(s)\mathrm{d}\vec{s}.
\end{aligned}
$$

In the more general case, we can use the same argument: Whenever we add a new square Q_n to the set $\Omega_{n-1} = \dot\bigcup_{i=1,\dots,n-1} Q_i$, we can call the part of the boundary where the two sets touch a. Since both curves are integrated counter-clockwise, Ω_{n-1} contributes $\int_a V(s)\mathrm{d}\vec{s}$ to the total integral, and Q_n contributes $-\int_a V(s)\mathrm{d}\vec{s}$. Thus, the two contributions always cancel each other out, leading to the desired result. All other parts of the boundaries of Ω_{n-1} and Q_n combine to form the boundary of Ω_n. Note that its not necessary that a is exactly one whole side of the square Q_n — it can also be more sides or only part of one side.

2. Consider the energies of regions Ω_1 and Ω_2 *before* and *after* the merge operation:

$$
\begin{aligned}
E_{\text{before}} &= \int_{\Omega_1} (I(x) - u_1)^2 \mathrm{d}x + \int_{\Omega_2} (I(x) - u_2)^2 \mathrm{d}x + \nu |C_{\text{before}}| \\
E_{\text{after}} &= \int_{\Omega_1 \cup \Omega_2} (I(x) - u_{\text{merged}})^2 \mathrm{d}x + \nu |C_{\text{after}}|.
\end{aligned}
$$

Here we assume that u_1, u_2 and u_{merged} optimize the energy given the respective region boundaries, i.e. they are the average intensity of the respective region (shown in the lecture). From this it follows that

$$
u_{\text{merged}} = \frac{u_1 A_1 + u_2 A_2}{A_1 + A_2}, \tag{1}
$$

which means u_{merged} is a weighted average of u_1 and u_2.

Furthermore we are going to use the fact that for the average $\bar{f}$ of a function f on a domain Ω,

$$\begin{aligned}\int_\Omega (f(x)-\bar{f})^2 \mathrm{d}x &= \int_\Omega f(x)^2 \mathrm{d}x - 2\bar{f}\int_\Omega f(x)\mathrm{d}x + \bar{f}^2 \int_\Omega \mathrm{d}x \\ &= \int_\Omega f(x)^2\mathrm{d}x - 2\bar{f}|\Omega|\bar{f} + \bar{f}^2|\Omega| = \int_\Omega f(x)^2\mathrm{d}x - |\Omega|\overline{f}^2,\end{aligned} \tag{2}$$

which is true in particular for $f = I$, $\bar{f} = u_i$ and $\Omega = \Omega_i$.

Since merging two regions always results in the contour C getting shorter, we can define a change $\delta C > 0$ in contour length as

$$\delta C = |C_{\text{after}}| - |C_{\text{before}}| \, .$$

For the change in energy δE, we adopt the more common definition of substracting the 'before'-value from the 'after'-value:

$$\begin{aligned}\delta E &= E_{\text{after}} - E_{\text{before}} \\ &= \int_{\Omega_1\cup\Omega_2} (I(x)-u_{\text{merged}})^2\mathrm{d}x - \int_{\Omega_1}(I(x)-u_1)^2\mathrm{d}x - \int_{\Omega_2}(I(x)-u_2)^2\mathrm{d}x - \nu\delta C \\ &= \int_{\Omega_1\cup\Omega_2} I(x)^2\mathrm{d}x - (A_1+A_2)u_{\text{merged}}^2 && \text{(using (2))} \\ &\quad - \int_{\Omega_1} I(x)^2\mathrm{d}x + A_1{u_1}^2 - \int_{\Omega_2} I(x)^2\mathrm{d}x + A_2{u_2}^2 - \nu\delta C \\ &= A_1{u_1}^2 + A_2{u_2}^2 - (A_1+A_2)\left(\frac{u_1A_1+u_2A_2}{A_1+A_2}\right)^2 - \nu\delta C && \text{(using (1))} \\ &= A_1{u_1}^2 + A_2{u_2}^2 - \frac{(u_1A_1+u_2A_2)^2}{A_1+A_2} - \nu\delta C \\ &= A_1{u_1}^2 + A_2{u_2}^2 - \frac{(u_1A_1)^2 + 2u_1A_1u_2A_2 + (u_2A_2)^2}{A_1+A_2} - \nu\delta C \\ &= \frac{(A_1+A_2)A_1{u_1}^2 + (A_1+A_2)A_2{u_2}^2 - (u_1A_1)^2 - 2u_1A_1u_2A_2 - (u_2A_2)^2}{A_1+A_2} - \nu\delta C \\ &= \frac{A_1A_2{u_1}^2 + A_1A_2{u_2}^2 - 2A_1A_2u_1u_2}{A_1+A_2} - \nu\delta C \\ &= \frac{A_1A_2}{A_1+A_2}(u_1-u_2)^2 - \nu\delta C.\end{aligned}$$