Variational Methods for Computer Vision: Solution Sheet 6

Exercise: December 5, 2017

Part I: Theory

1. Reminder: a linear map from X to Y is a function $L : X \to Y$ with the following properties

   (a) $L(u+v) = Lu + Lv, \;\; \forall u, v \in X;$

   (b) $L(\alpha u) = \alpha(Lu), \;\; \forall u \in X, \alpha \in \mathbb{R}.$

   As the $\tilde{e}_i$ form a basis, we can write Le_k uniquely as

   $$Le_k = M_{1,k}\tilde{e}_1 + \ldots + M_{m,k}\tilde{e}_m,$$

   for all $k \in \{1, \cdots, n\}$. These scalars $M_{i,j}$ then completely determine the linear map L. The matrix

   $$M = \begin{pmatrix} M_{1,1} & \cdots & M_{1,n} \\ \vdots & \ddots & \vdots \\ M_{m,1} & \cdots & M_{m,n} \end{pmatrix} = \begin{pmatrix} | & & | \\ [L(e_1)]_{\tilde{e}} & \cdots & [L(e_n)]_{\tilde{e}} \\ | & & | \end{pmatrix} \in \mathbb{R}^{m \times n}$$

   is then the so called *matrix representation* of L with respect to the bases $\{e_1, \cdots, e_n\}$ and $\{\tilde{e}_1, \cdots, \tilde{e}_m\}$.

   We verify that for some $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$ we have

   $$[L(x)]_{\tilde{e}} = \sum_{j=1}^{n} \alpha_j [L(e_j)]_{\tilde{e}} = \sum_{j=1}^{n} \alpha_j M_{\cdot,j} = Mx.$$

2. We start by the directional derivative, as on the last sheets:

   $$\begin{aligned} \left.\frac{\mathrm{d}E(u)}{\mathrm{d}u}\right|_h &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(E(u+\varepsilon h) - E(u)\right) \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\Omega \left(\mathcal{L}(u+\varepsilon h, \nabla(u+\varepsilon h), A(u+\varepsilon h)) - \mathcal{L}(u, \nabla u, Au)\right) \,\mathrm{d}x \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\Omega \left(\mathcal{L}(u+\varepsilon h, \nabla u + \varepsilon \nabla h, Au + \varepsilon Ah) - \mathcal{L}(u, \nabla u, Au)\right) \,\mathrm{d}x \\ &= \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u} h + \frac{\partial \mathcal{L}}{\partial \nabla u} \nabla h + \frac{\partial \mathcal{L}}{\partial Au}(Ah)\right) \,\mathrm{d}x \\ &= \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u} h - \operatorname{div} \frac{\partial \mathcal{L}}{\partial \nabla u} h + A^* \frac{\partial \mathcal{L}}{\partial Au} h\right) \,\mathrm{d}x + \int_{\partial\Omega} h \left\langle \frac{\partial \mathcal{L}}{\partial \nabla u}, n \right\rangle \mathrm{d}s \end{aligned}$$

   Hence the Euler-Lagrange equation is:

   $$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial u} - \operatorname{div} \frac{\partial \mathcal{L}}{\partial \nabla u} + A^* \frac{\partial \mathcal{L}}{\partial Au} = 0, \text{ for } x \in \Omega \\ &\left\langle \frac{\partial \mathcal{L}}{\partial \nabla u}, n \right\rangle = 0, \text{ for } x \in \partial\Omega, n \text{ is the normal.} \end{aligned}$$