

# Variational Methods for Computer Vision: Solution Sheet 5

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Exercise: November 28, 2017

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## Part I: Theory

1. (a)

$$\frac{dE_{\text{Poisson}}}{du} = -\operatorname{div} \left( \frac{\partial \mathcal{L}}{\partial(\nabla u)} \right) = -2 \operatorname{div} (\nabla u - V) = -2 (\Delta u - \operatorname{div} V) = -2 (\Delta u - \Delta g).$$

(b) Define a difference function  $d$  by  $d(x) := u(x) - g(x)$ . Since differentiation is a linear operation, we can write

$$J_u - J_g = J_d, \quad \nabla u_c - \nabla g_c = \nabla d_c \quad \text{for } c \in \{r, g, b\}$$

$$\begin{aligned} \Rightarrow \|J_u - J_g\|_F^2 &= \|J_d\|_F^2 = \left\| \begin{pmatrix} \partial_x d_r & \partial_y d_r \\ \partial_x d_g & \partial_y d_g \\ \partial_x d_b & \partial_y d_b \end{pmatrix} \right\|_F^2 \\ &= \sum_{c=r,g,b} (\partial_x d_c)^2 + (\partial_y d_c)^2 = \sum_{c=r,g,b} |\nabla d_c|^2 = \sum_{c=r,g,b} |\nabla u_c - \nabla g_c|^2 \end{aligned}$$

Now we can easily conclude that

$$E_{\text{P,vec}}(u) = \int_{\Omega_0} \|J_u - J_g\|_F^2 dx = \sum_{c=r,g,b} \int_{\Omega_0} |\nabla u_c - \nabla g_c|^2 dx = \sum_{c=r,g,b} E_{\text{Poisson}}(u_c)$$

2. We can write the energy functional as a sum of two energies

$$E(u) = E_k(u) + \lambda E_{\text{TV}}(\nabla u) \quad \Rightarrow \quad \frac{dE(u)}{du} = \frac{dE_k(u)}{du} + \lambda \frac{dE_{\text{TV}}(u)}{du}$$

where we know the second term of the Gateaux derivative of  $E$  from the last exercise sheet:

$$\frac{dE_{\text{TV}}}{du} = -\frac{\partial \mathcal{L}_{\text{TV}}(\nabla u)}{\partial(\nabla u)} = -\operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right),$$

so we can focus on computing the first term:

$$\begin{aligned} \left. \frac{dE_k}{du} \right|_h &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} \left( \frac{1}{2} (k * (u + \varepsilon h) - f)^2 - \frac{1}{2} (k * u - f)^2 \right) dx \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} \left( \frac{1}{2} (k * u - f + \varepsilon k * h)^2 - \frac{1}{2} (k * u - f)^2 \right) dx \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Omega} \left( \frac{1}{2} (k * u - f)^2 + \varepsilon (k * u - f)(k * h) + \frac{1}{2} \varepsilon^2 (k * h)^2 - \frac{1}{2} (k * u - f)^2 \right) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left( (k * u - f)(k * h) + \frac{1}{2} \varepsilon (k * h)^2 \right) dx = \int_{\Omega} (k * u - f)(k * h) dx. \end{aligned}$$

Now note that for convolution it holds that

$$\begin{aligned} \int g(x)(k * h)(x)dx &= \iint g(x)k(x-y)h(y)dydx \\ &= \iint g(x)\bar{k}(y-x)h(y)dx dy = \int (g * \bar{k})(y)h(y)dy = \int (g * \bar{k})(x)h(x)dx, \end{aligned}$$

where we defined  $\bar{k}(x) := k(-x)$ , so we obtain (by setting  $g := k * u - f$ )

$$\begin{aligned} \left. \frac{dE_k}{du} \right|_h &= \int_{\Omega} (k * u - f)(k * h)dx = \int_{\Omega} ((k * u - f) * \bar{k}) h dx \\ &\Rightarrow \frac{dE_k(u)}{du} = (k * u - f) * \bar{k} \end{aligned}$$

Thus the Euler-Lagrange equations for the functional

$$E(u) = \frac{1}{2} \int_{\Omega} (u * k - f)^2 dx + \lambda \int_{\Omega} |\nabla u| dx,$$

are given as follows:

$$\begin{aligned} (u * k - f) * \bar{k} - \lambda \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right) &= 0, & \text{in } \Omega, \\ \left\langle \frac{\nabla u}{|\nabla u|}, n \right\rangle &= 0, & \text{on } \partial\Omega. \end{aligned}$$