Variational Methods for Computer Vision: Solution Sheet 4

Exercise: November 21, 2017

Part I: Theory

1. In the one-dimensional case we write $\Omega = [a, b] \subset \mathbb{R}$, meaning that the boundary $\partial\Omega = \{a, b\}$. We proceed accordingly to the lecture:

$$
\begin{aligned}
\frac{\mathrm{d}E(u)}{\mathrm{d}u}\Big|_h &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(E(u+\varepsilon h) - E(u)\right) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_a^b \mathcal{L}(u+\varepsilon h, u'+\varepsilon h', u''+\varepsilon h'') - \mathcal{L}(u, u', u'')\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_a^b \mathcal{L}(u, u', u'') + \frac{\partial\mathcal{L}}{\partial u}\varepsilon h + \frac{\partial\mathcal{L}}{\partial u'}\varepsilon h' + \frac{\partial\mathcal{L}}{\partial u''}\varepsilon h'' + \mathcal{O}(\varepsilon^2) - \mathcal{L}(u, u', u'')\,\mathrm{d}x \\
&= \int_a^b \frac{\partial\mathcal{L}}{\partial u}h + \frac{\partial\mathcal{L}}{\partial u'}h' + \frac{\partial\mathcal{L}}{\partial u''}h''\,\mathrm{d}x \\
&= \int_a^b \frac{\partial\mathcal{L}}{\partial u}h\,\mathrm{d}x - \int_a^b h\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial u'}\,\mathrm{d}x - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial u''}h'\,\mathrm{d}x + \left[h\frac{\partial\mathcal{L}}{\partial u'}\right]_a^b + \left[h'\frac{\partial\mathcal{L}}{\partial u''}\right]_a^b \\
&= \int_a^b h\left(\frac{\partial\mathcal{L}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial u'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial\mathcal{L}}{\partial u''}\right)\mathrm{d}x - \left[h\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial u''}\right]_a^b + \left[h\frac{\partial\mathcal{L}}{\partial u'}\right]_a^b + \left[h'\frac{\partial\mathcal{L}}{\partial u''}\right]_a^b.
\end{aligned}
$$

2. (a)

$$
\begin{aligned}
\operatorname{div}(fg) &= \sum_{i=1}^n \frac{\partial(fg_i)}{\partial x_i} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}g_i + f\frac{\partial g_i}{\partial x_i}\right) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}g_i + f\sum_{i=1}^n \frac{\partial g_i}{\partial x_i} \\
&= \nabla f \cdot g + f \operatorname{div} g.
\end{aligned}
$$

   (b) By the divergence theorem, we know that $\int_\Omega \operatorname{div}(fg)\,\mathrm{d}x = \int_{\partial\Omega}\langle fg, n\rangle\,\mathrm{d}s$. Replacing the divergence with the result from question 2a yields

$$
\int_\Omega \langle\nabla f, g\rangle\,\mathrm{d}x + \int_\Omega f\operatorname{div} g\,\mathrm{d}x = \int_{\partial\Omega} f\langle g, n\rangle\,\mathrm{d}s,
$$

   which is easily transformed into the desired result,

$$
\int_\Omega \langle\nabla f, g\rangle\,\mathrm{d}x = \int_{\partial\Omega} f\langle g, n\rangle\,\mathrm{d}s - \int_\Omega f\operatorname{div} g\,\mathrm{d}x.
$$

3. To simplify notation, we write $(x\ y\ z)^\top = p \in \mathbb{R}^3$, $(u_x\ u_y\ u_z)^\top = \nabla u$ and

$$
\frac{\partial\mathcal{L}}{\partial\nabla u} = \left(\frac{\partial\mathcal{L}}{\partial u_x}\ \frac{\partial\mathcal{L}}{\partial u_y}\ \frac{\partial\mathcal{L}}{\partial u_z}\right)^\top.
$$

$$
\begin{aligned}
&\frac{\mathrm{d}E(u)}{\mathrm{d}u}\Big|_h = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(E(u+\varepsilon h)-E(u)\right)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \mathcal{L}(u+\varepsilon h, \nabla(u+\varepsilon h)) - \mathcal{L}(u,\nabla u)\,\mathrm{dp}\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega \mathcal{L}(u,\nabla u) + \frac{\partial\mathcal{L}}{\partial u}\varepsilon h + \frac{\partial\mathcal{L}}{\partial u_x}\varepsilon\frac{\partial h}{\partial x} + \frac{\partial\mathcal{L}}{\partial u_y}\varepsilon\frac{\partial h}{\partial y} + \frac{\partial\mathcal{L}}{\partial u_z}\varepsilon\frac{\partial h}{\partial z} + \mathcal{O}(\varepsilon^2) - \mathcal{L}(u,\nabla u)\,\mathrm{dp}\\
&= \int_\Omega \frac{\partial\mathcal{L}}{\partial u}h\,\mathrm{dp} + \int_\Omega \left\langle \nabla h, \left(\frac{\partial\mathcal{L}}{\partial u_x}\ \frac{\partial\mathcal{L}}{\partial u_y}\ \frac{\partial\mathcal{L}}{\partial u_z}\right)^T\right\rangle \mathrm{dp}\\
&= \int_\Omega \frac{\partial\mathcal{L}}{\partial u}h\,\mathrm{dp} - \int_\Omega h\,\mathrm{div}\left(\frac{\partial\mathcal{L}}{\partial u_x}\ \frac{\partial\mathcal{L}}{\partial u_y}\ \frac{\partial\mathcal{L}}{\partial u_z}\right)^T \mathrm{dp} + \int_{\partial\Omega} h\left\langle \left(\frac{\partial\mathcal{L}}{\partial u_x}\ \frac{\partial\mathcal{L}}{\partial u_y}\ \frac{\partial\mathcal{L}}{\partial u_z}\right)^T, n\right\rangle \mathrm{ds}\\
&= \int_\Omega h\left(\frac{\partial\mathcal{L}}{\partial u} - \left(\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial u_y} + \frac{\partial}{\partial z}\frac{\partial\mathcal{L}}{\partial u_z}\right)\right)\mathrm{dp} + \int_{\partial\Omega} h\left(\frac{\partial\mathcal{L}}{\partial u_x}n_x + \frac{\partial\mathcal{L}}{\partial u_y}n_y + \frac{\partial\mathcal{L}}{\partial u_z}n_z\right)\mathrm{ds}.
\end{aligned}
$$

Hence we can write the Euler-Lagrange equations as

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial u} - \mathrm{div}\left(\frac{\partial\mathcal{L}}{\partial\nabla u}\right) &= 0, && \text{on } \Omega,\\
\left\langle \frac{\partial\mathcal{L}}{\partial\nabla u}, n\right\rangle &= 0, && \text{on } \partial\Omega,
\end{aligned}
$$

where n denotes the normal vector on the boundary $\partial\Omega$.

Note that if the boundary term vanishes, we have

$$
\langle E(u), h\rangle = \frac{\mathrm{d}E(u)}{\mathrm{d}u}\Big|_h = \left\langle \frac{\partial\mathcal{L}}{\partial u} - \mathrm{div}\left(\frac{\partial\mathcal{L}}{\partial\nabla u}\right), h\right\rangle,\ \ \forall h,
$$

from which follows that

$$
E(u) = \frac{\partial\mathcal{L}}{\partial u} - \mathrm{div}\left(\frac{\partial\mathcal{L}}{\partial\nabla u}\right).
$$

Remark: To see the last equality, consider for scalar products in general

$$
\begin{aligned}
&\langle f, h\rangle = \langle g, h\rangle, \forall h\\
\Leftrightarrow\ &\langle f-g, h\rangle = 0, \forall h\\
\Rightarrow\ &\langle f-g, f-g\rangle = 0,\\
\Rightarrow\ &f = g.
\end{aligned}
$$

4. (a) Using the same notation as previously, we have that $\mathcal{L}(u,\nabla u) = \sqrt{u_x^2+u_y^2}$ and

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial u_x} &= \frac{u_x}{\sqrt{u_x^2+u_y^2}},\\
\frac{\partial\mathcal{L}}{\partial u_y} &= \frac{u_y}{\sqrt{u_x^2+u_y^2}},
\end{aligned}
$$

with which we evaluate

$$
\frac{\mathrm{d}E}{\mathrm{d}u} = \frac{\partial\mathcal{L}}{\partial u} - \mathrm{div}\left(\frac{\partial\mathcal{L}}{\partial\nabla u}\right) = -\,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right).
$$

Thus the Euler-Lagrange equations are given as the following:

$$\begin{aligned} \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) &= 0, && \text{on } \Omega, \\ \left\langle \frac{\nabla u}{|\nabla u|}, n \right\rangle &= 0, && \text{on } \partial\Omega. \end{aligned}$$

(b) Similarly to the previous exercise, by considering partial derivatives by u_x and u_y and expanding the matrix products, we derive for

$$\mathcal{L}(u, \nabla u) = \sqrt{(\nabla u)^\mathsf{T} D \nabla u} = \sqrt{(u_x \; u_y) \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}}$$

the partial derivative

$$\frac{\partial \mathcal{L}}{\partial \nabla u} = \left(\frac{\partial \mathcal{L}}{\partial u_x} \; \frac{\partial \mathcal{L}}{\partial u_y}\right)^\top = \cdots = \frac{(D + D^\mathsf{T})\nabla u}{2\sqrt{(\nabla u)^\mathsf{T} D \nabla u}},$$

and thus

$$\frac{\mathrm{d}E}{\mathrm{d}u} = \frac{\partial \mathcal{L}}{\partial u} - \operatorname{div}\left(\frac{\partial \mathcal{L}}{\partial \nabla u}\right) = -\operatorname{div}\left(\frac{(D + D^\mathsf{T})\nabla u}{2\sqrt{(\nabla u)^\mathsf{T} D \nabla u}}\right),$$

with which we arrive at the following Euler-Lagrange equations:

$$\begin{aligned} \operatorname{div}\left(\frac{(D + D^\mathsf{T})\nabla u}{2\sqrt{(\nabla u)^\mathsf{T} D \nabla u}}\right) &= 0, && \text{on } \Omega, \\ \left\langle \frac{(D + D^\mathsf{T})\nabla u}{2\sqrt{(\nabla u)^\mathsf{T} D \nabla u}}, n \right\rangle &= 0, && \text{on } \partial\Omega. \end{aligned}$$