Variational Methods for Computer Vision: Exercise Sheet 6

Exercise: December 5, 2017

Part I: Theory

1. Let $L : X \to Y$ be a linear operator and X, Y be finite dimensional vector spaces with dim $X = n$ and dim $Y = m$. Let $\{e_1, ..., e_n\}$ and $\{\tilde{e}_1, ..., \tilde{e}_m\}$ be the bases for X and respectively for Y. Show that the operator L can be represented by an $m \times n$ matrix M, hence:

$$L(u) = Mu, \qquad \forall u \in X.$$

2. Calculate the Euler-Lagrange equation of the following energy functional

$$E(u) = \int_\Omega \mathcal{L}(u(x), \nabla u(x), Au(x)) \,\mathrm{dx},$$

where $\Omega \subset \mathbb{R}^2$, $u : \Omega \to \mathbb{R}$, and $A : (\Omega \to \mathbb{R}) \to (\Omega \to \mathbb{R})$ is a linear mapping.

Hint: use the adjoint A^* of the operator A for which the following identity holds

$$\int_\Omega u(x)(Av)(x)\mathrm{d}x = \int_\Omega (A^* u)(x)v(x)\mathrm{d}x.$$

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

Super-Resolution from Video.
In the lecture we encountered the concept of super resolution from video. The key idea of super resolution is to exploit redundancy available in multiple frames of a video. Assuming that each input frame is a blurred and downsampled version of a higher resolved image u, the high-resolution image can be recovered as the minimum of the following energy functional:

$$E(u) = \sum_{i=1}^{n} \int_{\Omega} ((ABS_i u)(x) - (Uf_i)(x))^2 \, \mathrm{d}x + \lambda \int_{\Omega} |\nabla u(x)| \, \mathrm{d}x. \tag{1}$$

The Linear Operator B denotes a Gaussian Blurring. The upsampling operator U simply replaces every pixel with four pixels of the same intensity. In order to be able to compare image u with the upsampled version of f_i which is constant blockwise, we apply the linear averaging operator A on u which assigns every block of pixels the mean values of the pixels in that block. The linear operator S_i accounts for the coordinate shift by motion s_i hence:

$$(S_i u)(x) = u(x + s_i(x)).$$

1. In the following we are going to construct a toy example for super resolution by executing the following steps:

   (a) Download the archive `vmcv_ex06.zip` and unzip it on your home folder. In there should be a file named `Boat.png`.

   (b) Create from the unzipped image 6 versions shifted in x direction by exactly one pixel hence:
   $$f_i(x, y) = f(x + i, y),$$
   for $i = 1...6$. In order to account for the boundary, consider taking cropped images from the interior of the original image.

   (c) In order to simulate blurring convolve the shifted images with a gaussian kernel. Next downsample the images f_i by factor 2 by using the `imresize` function in Matlab with nearest neighbor interpolation.

2. In what follows we are going to minimize the above functional in order to obtain a super resolved image from our input images f_i.

   (a) Derive the Euler-Lagrange equation of E and the corresponding gradient descent scheme.

   (b) Compute the matrix representations of the linear operators A, B, S_i and U. Since these matrices are huge, again use sparse data structures in Matlab (`spdiags speye`) in order to obtain a sparse representation.

   (c) Compute $u^* = \arg\min_u E(u)$ by means of gradient descent using matrix vector representation after stacking the function u in a vector using the matlab command `reshape`.