

Weekly Exercises 2

Room: 02.05.014

Tuesday, 03.11.2015, 14:15-15:45

Submission deadline: Tuesday, 03.11.2015, 11:15 , Room 02.09.023

Probability Theory

(10 Points)

Exercise 1 (σ -Algebra, 3 Points).

a) Let $\mathcal{A} \subset \mathcal{P}(\Omega)$, such that

$$A \in \mathcal{A} \Rightarrow \Omega \setminus A \in \mathcal{A} \text{ and} \\ A_1, A_2, \dots \in \mathcal{A} \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}.$$

Show that

$$\emptyset \in \mathcal{A} \Leftrightarrow \mathcal{A} \neq \emptyset \Leftrightarrow \Omega \in \mathcal{A}.$$

b) Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ be a σ -algebra, show that \mathcal{A} is closed under intersections. I.e.

$$A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cap A_2 \in \mathcal{A}.$$

Exercise 2 (Bayes' Rule, 3 Points). Siegfried the ornithologist does a study on the green-speckled swallow. Since he has a huge collection of bird photographs he wants to find all images depicting a green-speckled swallow. Due to its distinctive features it is an easy task for Eduard, Siegfried's friend and computer vision scientist, to program a green-speckled swallow detector that marks all images containing such a bird. Unfortunately the detector does not work perfectly. If the image contains a green-speckled swallow the detector marks it correctly with a chance of 99.5%. If the image does not contain a green-speckled swallow the detector marks it correctly with a chance of 99.3%. The bird is also very rare: If we randomly draw an image from the collection, there is only an chance of 0.001% that the image contains a green-speckled swallow.

- Do a formal modeling of the experiment. How does the discrete probability space look like?
- What is the probability that a green-speckled swallow is on a given image, if the detector gives a positive answer? What is the probability that a green-speckled swallow is on a given image, if the detector gives a negative answer?

Exercise 3 (Probability, 4 Points). In order to express his gratitude, Siegfried invites Eduard to a pub for a couple of beers. There, they start playing a friendly game of darts.

The dart board is a perfect disk of radius 10cm. If a dart falls within 1cm of the center, 100 points are scored. If the dart hits the board between 1 and 3cm from the center, 50 points are scored, if it is at a distance of 3 to 5cm 25 points are scored and if it is further away than 5cm 10 points are scored.

As Siegfried and Eduard are both quite experienced dart players, they hit the dart board every time. Siegfried places the dart uniformly on the board.

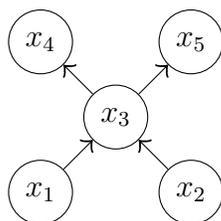
- Define the probability space (Ω, \mathcal{A}, P) and the random variable X .
- What is the probability that Siegfried scores 100 points on one throw?
- What is the probability of him scoring 50 points on one throw?
- Eduard is very focused and thus twice more likely to hit the inner 4cm part of the board than the outer region. On each region, the dart arrives uniformly. Answer the previous questions now for Eduard's throw.

Graphical Models (5 Points)

Exercise 4 (Conditional Independence, 2 Points). Consider four random variables a, b, c, d . Prove the following:

$$a \perp\!\!\!\perp (b, c) \mid d \text{ implies } a \perp\!\!\!\perp b \mid d.$$

Exercise 5 (Bayesian Network, 1 Point). Provide the the factorization $p(x_1, x_2, x_3, x_4, x_5)$ for the following directed graph:



Exercise 6 (MRF, 2 Points). Let G be a factor graph given by a Markov random field consisting of N^2 binary variables, representing the pixels of a $N \times N$ image. For each pixel there is a unary potential, and there are pairwise potentials according to the 4-connected neighbourhood.

- Draw the factor graph for $N = 3$.
- How many factors (of each type) are there, depending on N ?