Variational Methods for Computer Vision: Solution Sheet 6

Exercise: 5 December 2013

Part I: Theory

1. (a) The curvature κ of a circle with radius r is $\kappa = \frac{1}{r}$. We can use this fact in calculating the Euler-Lagrange equations for the 2 different cases.
   $r > 1$:

$$u_{\text{outer}} = 0$$
$$u_{\text{inner}} = \frac{\pi}{\pi r^2} = \frac{1}{r^2}$$

   This leads to following Euler-Lagrange equation:

$$\begin{aligned} &(I - u_{\text{outer}})^2 - (I - u_{\text{inner}})^2 - \nu\kappa \\ =&(0-0)^2 - (0 - \frac{1}{r^2})^2 - \frac{\nu}{r} \\ =&-\frac{1}{r^2} - \frac{\nu}{r} \end{aligned}$$

   $r \leq 1$:

$$u_{\text{outer}} = \frac{\pi - \pi r^2}{100 - \pi r^2}$$
$$u_{\text{inner}} = 1$$

   This leads to following Euler-Lagrange equation:

$$\begin{aligned} &(I - u_{\text{outer}})^2 - (I - u_{\text{inner}})^2 - \nu\kappa \\ =&\left(1 - \frac{\pi - \pi r^2}{100 - \pi r^2}\right)^2 - 0 - \frac{\nu}{r} \\ =&\frac{100 - \pi r^2 - \pi - \pi r^2}{100 - \pi r^2} - \frac{\nu}{r} \\ =&\left(\frac{100 - \pi}{100 - \pi r^2}\right)^2 - \frac{\nu}{r} \end{aligned}$$

   (b)

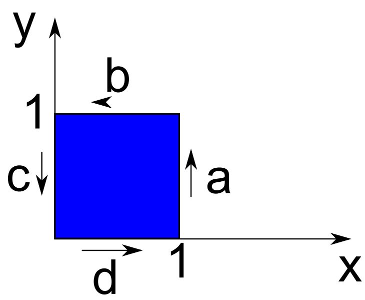
$$\lim_{r \searrow 1} -\frac{1}{r^2} - \frac{\nu}{r} = -1 - \nu$$
$$\lim_{r \nearrow 1} -\frac{100 - \pi}{100 - \pi r^2} - \frac{\nu}{r} = \frac{100 - \pi}{100 - \pi} - \nu = 1 - \nu$$

   As the limits differ the Gateux derivative at $r = 1$ is not continuous.
   $\nu \leq 1$ is a good choice because it ensures that the curve evolves in the right direction for both cases $r > 1$ and $r \leq 1$.
   $r > 1$:

$$\nu \leq 1 \Rightarrow -\frac{1}{r^2} - \frac{\nu}{r} < 0$$

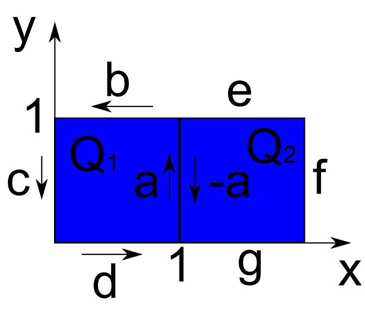
   $r \leq 1$:

$$\nu \leq 1 \Rightarrow \left(\frac{100 - \pi}{100 - \pi r^2}\right)^2 - \frac{\nu}{r} \geq 0$$



2. (a)

$$\begin{aligned}
\int\limits_Q v_x(x,y) - u_y(x,y)\mathrm{d}x\mathrm{d}y &= \int\limits_0^1 \int\limits_0^1 v_x(x,y) - u_y(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int\limits_0^1 \int\limits_0^1 v_x(x,y)\mathrm{d}x\mathrm{d}y - \int\limits_0^1 \int\limits_0^1 u_y(x,y)\mathrm{d}y\mathrm{d}x \\
&= \int\limits_0^1 v(x,y)|_{x=0}^{x=1}\,\mathrm{d}y - \int\limits_0^1 v(x,y)|_{y=0}^{y=1}\,\mathrm{d}x \\
&= \int\limits_0^1 v(1,y) - v(0,y)\mathrm{d}y - \int\limits_0^1 v(x,1) - v(x,0)\mathrm{d}x \\
&= \int\limits_0^1 v(1,y)\mathrm{d}y - \int\limits_0^1 v(0,y)\mathrm{d}y - \int\limits_0^1 v(x,1)\mathrm{d}x + \int\limits_0^1 v(x,0)\mathrm{d}x \\
&= \underbrace{\int\limits_0^1 v(1,y)\mathrm{d}y}_{\int_a v(x,y)\mathrm{d}y} + \underbrace{\int\limits_1^0 v(0,y)\mathrm{d}y}_{\int_c v(x,y)\mathrm{d}y} + \underbrace{\int\limits_1^0 v(x,1)\mathrm{d}x}_{\int_b u(x,y)\mathrm{d}x} + \underbrace{\int\limits_0^1 v(x,0)\mathrm{d}x}_{\int_d v(x,y)\mathrm{d}x} \\
&= \int\limits_{\partial Q} v\mathrm{d}s
\end{aligned}$$



(b)

$$\begin{aligned}
&\int\limits_{Q_1} v_x(x,y) - u_y(x,y)\mathrm{d}x\mathrm{d}y + \int\limits_{Q_2} v_x(x,y) - u_y(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int\limits_a v(x,y)\mathrm{d}y \int\limits_c v(x,y)\mathrm{d}y + \int\limits_b v(x,y)\mathrm{d}x + \int\limits_d v(x,y)\mathrm{d}x \\
&- \int\limits_a v(x,y)\mathrm{d}y + \int\limits_e v(x,y)\mathrm{d}x + \int\limits_f v(x,y)\mathrm{d}y + \int\limits_g v(x,y)\mathrm{d}x
\end{aligned}$$