Variational Methods for Computer Vision: Solution Sheet 4

Exercise: 21 November 2013

Part I: Theory

1.

$$
\begin{aligned}
\left.\frac{\partial E(u)}{\partial u}\right|_h &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}(E(u+\epsilon h)-E(u))\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int(\mathcal{L}(u+\epsilon h, u'+\epsilon h', u''+\epsilon h'')-\mathcal{L}(u,u',u''))\mathrm{d}x\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int(\mathcal{L}(u,u',u'')+\frac{\partial\mathcal{L}}{\partial u}\epsilon h+\frac{\partial\mathcal{L}}{\partial u'}\epsilon h'+\frac{\partial\mathcal{L}}{\partial u''}\epsilon h''+\mathcal{O}(\epsilon^2)-\mathcal{L}(u,u',u''))\mathrm{d}x\\
&= \int\left(\frac{\partial\mathcal{L}}{\partial u}h+\frac{\partial\mathcal{L}}{\partial u'}h'+\frac{\partial\mathcal{L}}{\partial u''}h''\right)\mathrm{d}x\\
&= \int\left(\frac{\partial\mathcal{L}}{\partial u}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial u'}+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial\mathcal{L}}{\partial u''}\right)h(x)\mathrm{d}x+\left(h(x)\frac{\partial\mathcal{L}}{\partial u'}\right)_a^b+\left(h'(x)\frac{\partial\mathcal{L}}{\partial u''}\right)_a^b-\left(h(x)\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\mathcal{L}}{\partial u''}\right)_a^b
\end{aligned}
$$

2.

$$
\begin{aligned}
\left.\frac{\partial E(u)}{\partial u}\right|_h &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}(E(u+\epsilon h)-E(u))\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int(\mathcal{L}(u+\epsilon h,\nabla(u+\epsilon h))-\mathcal{L}(u,\nabla u))\,\mathrm{d}x\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int\left(\mathcal{L}(u,\nabla u)+\frac{\partial\mathcal{L}}{\partial u}\epsilon h+\frac{\partial\mathcal{L}}{\partial u_x}\epsilon\frac{\partial}{\partial x}h+\frac{\partial\mathcal{L}}{\partial u_y}\epsilon\frac{\partial}{\partial y}h+\frac{\partial\mathcal{L}}{\partial u_z}\epsilon\frac{\partial}{\partial z}h-\mathcal{L}(u,\nabla u)\right)\mathrm{d}x\\
&= \int\left(\frac{\partial\mathcal{L}}{\partial u}h+\frac{\partial\mathcal{L}}{\partial u_x}\frac{\partial}{\partial x}h+\frac{\partial\mathcal{L}}{\partial u_y}\frac{\partial}{\partial y}h+\frac{\partial\mathcal{L}}{\partial u_z}\frac{\partial}{\partial z}h\right)\mathrm{d}x\\
&= \int\left(\frac{\partial\mathcal{L}}{\partial u}+\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial u_x}+\frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial u_y}+\frac{\partial}{\partial z}\frac{\partial\mathcal{L}}{\partial u_z}\right)h\mathrm{d}x+\int_{\partial\Omega}h\cdot\left[\frac{\partial\mathcal{L}}{\partial u_x},\frac{\partial\mathcal{L}}{\partial u_y},\frac{\partial\mathcal{L}}{\partial u_z}\right]\cdot\vec{n}
\end{aligned}
$$

Remark: $\vec{n}$ is the normal to the respective image boundary.

3. (a)

$$E_1=\int_\Omega|\nabla u|\mathrm{d}x=\int_\Omega\sqrt{\nabla u^T\nabla u}\mathrm{d}x=\int_\Omega\sqrt{{u_x}^2+{u_y}^2}\mathrm{d}x$$

Thus:

$$
\begin{aligned}
\frac{\partial|\nabla u|}{\partial u_x} &= 2u_x\frac{1}{2}({u_x}^2+{u_y}^2)^{-\frac{1}{2}}\\
&= \frac{u_x}{\sqrt{{u_x}^2+{u_y}^2}}\\
&= \frac{u_x}{|\nabla u|}
\end{aligned}
$$

And:

$$\begin{aligned}\frac{\partial |\nabla u|}{\partial u_y} &= 2u_y \frac{1}{2}({u_x}^2 + {u_y}^2)^{-\frac{1}{2}} \\ &= \frac{u_y}{\sqrt{{u_x}^2 + {u_y}^2}} \\ &= \frac{u_y}{|\nabla u|}\end{aligned}$$

As seen in the lecture the Euler-Lagrange equation for functionals in canonical form is

$$\frac{\partial E}{\partial u} = \frac{\partial \mathcal{L}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial u'}$$

In this special case $\frac{\partial \mathcal{L}}{\partial u} = 0$ holds. Using the result of exercise 2 one gets:

$$\begin{aligned}\frac{\partial E_1}{\partial u} = -\operatorname{div}\frac{\partial |\nabla u|}{\partial \nabla u} &= -\operatorname{div}\begin{pmatrix}\frac{\partial |\nabla u|}{\partial u_x} \\ \frac{\partial |\nabla u|}{\partial u_y}\end{pmatrix} \\ &= -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\end{aligned}$$

(b) Remark:

$$\frac{\partial}{\partial x}\left(x^T A x\right) = \left(A + A^T\right) x$$

Thus:

$$\begin{aligned}\frac{\partial E_2}{\partial u} &= \left(A + A^T\right) u \cdot \frac{1}{2\sqrt{\nabla u^T D \nabla u}} \\ &= \frac{\left(A + A^T\right) u}{2\sqrt{\nabla u^T D \nabla u}}\end{aligned}$$