geodesics family. We are currently investigating this geodesic-type approach for other problems in image analysis, as well as the use of better image metrics to incorporate into the geodesic model. These metrics, together with multi-scale implementations as in (Geiger et al., 1995) and fast numerical algorithms as those in (Adalsteinsson and Sethian, 1993), will improve possible initialization difficulties as those in Fig. 6 as well as performance speed.

The formulation of 3D active surfaces is an important topic for many applications as well; see for example (Cohen et al., 1992). Extension of the 2D curve evolution model developed in (Caselles et al., 1993; Malladi et al., 1994, 1995, —) is not straightforward, since an extension of the Euclidean heat flow was not yet developed (Alvarez et al., 1993; Caselles and Sbert, 1994; Olver et al., 1996). The geodesic formulation given by (8) can be extended to 3D replacing the 2D gradient by a 3D one and Euclidean arc-length (ds) by area. Then, using the level-sets representation, the corresponding geometric flow can be computed. Results in this direction are reported in (Caselles et al., 1995).

Appendix A

Let us present the analogue to Eq. (7) when E_0 is a general value. Note that E_0 gives the difference between E_{int} and E_{ext} in (2). If $E_0 \neq 0$, then instead of (7), the following minimization is obtained:

$$\text{Min} \int_0^1 \sqrt{2m}\sqrt{E_0 + \lambda g(I)^2}|C'|\,dq. \qquad \text{(A1)}$$

In order for all the computations after Eq. (7) to hold, the expression above is equivalent to (7) if

$$g \leftarrow \sqrt{2m}\sqrt{E_0 + \lambda g(I)^2}.$$

As pointed out before, E_0 represents the trade-off between α and λ in (2) (as well as the parametrization), as is clear from the expressions above. Let us further develop this point here for completeness.

Re-writing $E_0 + \lambda g^2(I)$ as a quadratic form $(\sqrt{E_0} + Q)^2$, it is easy to show that $Q = -\sqrt{E_0} + \sqrt{E_0 + \lambda g(I)^2}$ and (A1) becomes

$$\text{Min}\left(\int_0^1 Q\,ds + \sqrt{E_0}L\right),$$

where L is the Euclidean length of the curve. Since Q is an edge detector as g, we see that basically the minimization problem has an extra term related to the length of the curve. The importance of this length in the minimization is given by the exact value of E_0, manifesting the relation between E_0 and the trade-off parameters α and λ in the energy expression (2). Note that as explained before, the Euler-Lagrange of L is κ, and this will appear as an extra term in the corresponding flow if $E_0 \neq 0$. Then, the new geodesic flow will be (compare with (13))

$$\frac{\partial \mathcal{C}(t)}{\partial t} = Q(I)\kappa\vec{\mathcal{N}} - (\nabla Q \cdot \vec{\mathcal{N}})\vec{\mathcal{N}} + \sqrt{E_0}\kappa\vec{\mathcal{N}} \qquad \text{(A2)}$$

The extra term appears un-related to Q, which is the edge detector part of the algorithm. Therefore, selecting E_0 too big, will give too much importance to the minimization of L, and may cause the flow to miss the edges. This is clear also from (2), which (A2) is trying to minimize. Having $E_0 = 0$ is the only option which makes all the components of the geometric flow that minimizes (2) to be g-dependent, giving a further justification for this selection.

Appendix B

We now compute the Euler-Lagrange of (8), to obtain the geodesic flow (13). For the simplification of the notation, we sometimes write $\mathcal{C}(t)$ for the curve $\mathcal{C}(t, q)$, omitting the space parameter q, as well as $g(\mathcal{C})$ instead of $g(|\nabla I(\mathcal{C})|)$.

Consider the functional

(0) $$L_R(\mathcal{C}) = \int_0^1 g(\mathcal{C}(t,q))|\mathcal{C}_q(t,q)|\,dq,$$

where $\mathcal{C}$: $[0, 1] \to \mathbf{R}^2$ is a closed (C^1) curve. Let us compute the first variation of L_R at some closed curve $\mathcal{C}_0$, assumed to be of class C^2. Consider a variation $\mathcal{C}$ of $\mathcal{C}_0$, that is

$$\mathcal{C}: (-\epsilon, \epsilon) \times [0, 1] \to \mathbf{R}^2$$
$$(t, q) \to \mathcal{C}(t, q),$$

is a C^2 function of (t, q) such that $\mathcal{C}(0, q) \equiv \mathcal{C}_0$ and $\mathcal{C}(t, 0) = \mathcal{C}(t, 1)$, $t \in (-\epsilon, \epsilon)$ $(\epsilon > 0)$. Assuming a given orientation of $\mathcal{C}$, we compute the derivative of $L_R(\mathcal{C})$ with respect of t, obtaining

(1) $$\frac{d}{dt}L_R(\mathcal{C}(t)) = \int_0^1 \frac{d}{dt}g(\mathcal{C}(t,q))|\mathcal{C}_q(t,q)|\,dq + \int_0^1 g(\mathcal{C}(t,q))\frac{d}{dt}|\mathcal{C}_q(t,q)|\,dq.$$

Therefore,

(2) $$\frac{d}{dt}L_R(\mathcal{C}(t)) = \int_0^1 (\nabla g(\mathcal{C}(t,q)) \cdot \mathcal{C}_t(t,q))|\mathcal{C}_q(t,q)|\,dq + \int_0^1 g(\mathcal{C}(t,q))(\vec{\mathcal{T}}(t,q) \cdot \mathcal{C}_{tq}(t,q))\,dq,$$

where $\vec{\mathcal{T}}(t, q)$ denotes the unit tangent to the curve $\mathcal{C}(t, q)$. Integrating by parts in the second term we have that the above expression is equal to

(3) $$= \int_0^1 (\nabla g(\mathcal{C}(t,q)) \cdot \mathcal{C}_t(t,q))|\mathcal{C}_q(t,q)|\,dq - \int_0^1 (g(\mathcal{C}(t,q))\vec{\mathcal{T}}(t,q))_q \cdot \mathcal{C}_t(t,q))\,dq$$

(4) $$= \int_0^1 [(\nabla g(\mathcal{C}(t,q)) \cdot \mathcal{C}_t(t,q))|\mathcal{C}_q(t,q)| - (\nabla g(\mathcal{C}(t,q)) \cdot \mathcal{C}_q(t,q))(\vec{\mathcal{T}}(t,q) \cdot \mathcal{C}_t(t,q)) - g(\mathcal{C}(t,q))\vec{\mathcal{T}}_q(t,q) \cdot \mathcal{C}_t(t,q)]\,dq$$

(5) $$= \int_0^1 [((\nabla g(\mathcal{C}(t,q)) \cdot \mathcal{C}_t(t,q) - (\nabla g(\mathcal{C}(t,q)) \cdot \vec{\mathcal{T}}(t,q)) \cdot (\vec{\mathcal{T}}(t,q) \cdot \mathcal{C}_t(t,q)))|\mathcal{C}_q(t,q)| - g(\mathcal{C}(t,q))\vec{\mathcal{T}}_q(t,q) \cdot \mathcal{C}_t(t,q)]\,dq.$$

Let s denote the arc-length of $\mathcal{C}(t)$. Since $\vec{\mathcal{T}}_q = \vec{\mathcal{T}}_s|\mathcal{C}_q|$, parametrizing the curves by arc-length, the above integral writes

(6) $$\int_0^{L(\mathcal{C}(t))} [(\nabla g(\mathcal{C}(t,s)) \cdot \mathcal{C}_t(t,s)) - (\nabla g(\mathcal{C}(t,s)) \cdot \vec{\mathcal{T}}(t,s)) (\vec{\mathcal{T}}(t,s) \cdot \mathcal{C}_t(t,s)) - g(\mathcal{C}(t,s))\vec{\mathcal{T}}_s(t,s) \cdot \mathcal{C}_t(t,s)]\,ds.$$

To simplify the notation let us remove the arguments in the expression above, obtaining

(7) $$\frac{d}{dt}L_R(\mathcal{C}(t)) = \int_0^{L(\mathcal{C}(t))} [\nabla g(\mathcal{C}) - (\nabla g(\mathcal{C}) \cdot \vec{\mathcal{T}})\vec{\mathcal{T}} - g(\mathcal{C})\vec{\mathcal{T}}_s] \cdot \mathcal{C}_t\,ds.$$

At $t = 0$,

(8) $$\frac{d}{dt}L_R(\mathcal{C}(t))|_{t=0} = \int_0^{L(\mathcal{C}_0)} [\nabla g(\mathcal{C}_0) - (\nabla g(\mathcal{C}_0) \cdot \vec{\mathcal{T}})\vec{\mathcal{T}} - g(\mathcal{C}_0)\vec{\mathcal{T}}_s] \cdot \mathcal{C}_t(0)\,ds.$$

Since $\vec{\mathcal{T}}_s = \kappa\vec{\mathcal{N}}$, we have

(9) $$\frac{d}{dt}L_R(\mathcal{C}(t))|_{t=0} = \int_0^{L(\mathcal{C}_0)} [\nabla g(\mathcal{C}_0) - (\nabla g(\mathcal{C}_0) \cdot \vec{\mathcal{T}})\vec{\mathcal{T}} - g(\mathcal{C}_0)\kappa\vec{\mathcal{N}}] \cdot \mathcal{C}_t(0)\,ds,$$

and

(10) $$\frac{d}{dt}L_R(\mathcal{C}(t))|_{t=0} = \int_0^{L(\mathcal{C}_0)} [(\nabla g(\mathcal{C}_0) \cdot \vec{\mathcal{N}})\vec{\mathcal{N}} - g(\mathcal{C}_0)\kappa\vec{\mathcal{N}}] \cdot \mathcal{C}_t(0)\,ds.$$

This expression gives the Gateaux derivative (first variation) of L_R at $\mathcal{C} = \mathcal{C}_0$. Then, according to the steepest-descent method, to connect an initial curve $\mathcal{C}_0$ with a local minimum of $L_R(\mathcal{C})$ we should solve the evolution equation

(11) $$\mathcal{C}_t = g(\mathcal{C})\kappa\vec{\mathcal{N}} - (\nabla g(\mathcal{C}) \cdot \vec{\mathcal{N}})\vec{\mathcal{N}}.$$

This gives (13), that is, the motion of the level-sets of (13), minimizing (8). To compute the motion of the embedding function u, the results in next Appendix are used. Following the same steps as before, it can also be shown that (13) is the flow corresponding to the steepest-descent of

$$E(u) = \int_{\mathbf{R}^2} g(\mathcal{X})|\nabla u|\,d\mathcal{X}.$$

Appendix C

We present a geometric result concerning the evolution of the embedding function u given the flow of its level-sets.

Consider a planar curve evolving according to

$$\mathcal{C}_t = \beta\vec{\mathcal{N}},$$

for a given function β. We want to represent $\mathcal{C}$ as the level-set of a function u: $\mathbf{R}^2 \to \mathbf{R}$. The question is how u should evolve. This embedding process was first proposed in the curve evolution framework in (Osher and Sethian, 1988), and we proceed to give a very simple geometric derivation of it. Formal justification of the method, on the lines described in Section 3, was later provided in (Chen et al., 1991; Evans and Spruck, 1991; Soner, 1993). Assume that u is negative in the interior