Part I: Theory

1. Show for each of the following sets (1) whether they are linearly independent, (2) whether they span $\mathbb{R}^3$ and (3) whether they form a basis of $\mathbb{R}^3$:

(a) $B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

The set $B_1$ (1) is linearly independent, (2) spans $\mathbb{R}^3$, (3) forms a basis of $\mathbb{R}^3$.

This can be shown by building a matrix and calculating the determinant:

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 1 \neq 0.$$ 

As the determinant is not zero, we know that the vectors are linearly independent. Three linearly independent vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$. A set is a basis of $\mathbb{R}^3$ if it is linearly independent and spans $\mathbb{R}^3$, so $B_1$ forms a basis.

(b) $B_2 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

The set $B_2$ (1) is linearly independent, (2) does not span $\mathbb{R}^3$, (3) does not form a basis of $\mathbb{R}^3$.

Since the two vectors are not parallel, linear independence is given. To span $\mathbb{R}^3$, there are at least three vectors needed. Hence, the set cannot be a basis either.

(c) $B_3 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

The set $B_3$ (1) is not linearly independent, (2) spans $\mathbb{R}^3$, (3) does not form a basis of $\mathbb{R}^3$.

In $\mathbb{R}^3$, there cannot be more than three independent vectors. Using e.g. the determinant, one finds that any three of the four vectors form a basis of $\mathbb{R}^3$ and thus the four together span $\mathbb{R}^3$. Since they are not linearly independent, they cannot form a basis.

2. Which of the following sets forms a group (with matrix-multiplication)? Prove or disprove!

(a) $G_1 := \left\{ A \in \mathbb{R}^{n \times n} \mid \det(A) \neq 0 \land A^T = A \right\}$

The set is not closed under multiplication, thus no group. To show this, one counter-example is enough: choose $n = 3$ and

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{pmatrix} \in G_1, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in G_1: \quad AB = \begin{pmatrix} 1 & 4 & 9 \\ 2 & 0 & 12 \\ 3 & 8 & 15 \end{pmatrix} \notin G_1$$
Note:
You can also show that if $G_1$ was a group, for any $A, B \in G_1$, $(AB)^\top = AB$ would have to be true, but is not. This is equivalent to saying $BA = AB$ would have to be true:

$$(AB)^\top = B^\top A^\top = BA$$

However, to show that there exist $A$ and $B$ in $G_1$ for which $AB \neq BA$ (which is an important step in the proof!), the easiest way again is to choose a concrete counter-example.

(b) $G_2 := \{ A \in \mathbb{R}^{n \times n} | \det(A) = -1 \}$

The set contains no neutral element, thus no group:

$$\det(\text{Id}_n) = 1 \neq -1 \implies \text{Id}_n \notin G_2$$

(c) $G_3 := \{ A \in \mathbb{R}^{n \times n} | \det(A) > 0 \}$

The set forms a group. The easiest way to show this is to show that $G_3$ is a subgroup of the general linear group $GL(n)$. We simply need to show that for any two elements $A, B$ of $G_3$, $AB^{-1}$ is also in $G_3$:

$$\det(AB^{-1}) = \det(A) \frac{1}{\det(B)} > 0 \Rightarrow AB^{-1} \in G_3$$

Thus, $G_3$ is a subgroup of $GL(n)$ and hence a group.

3. Prove or disprove: There exist vectors $v_1, ..., v_5 \in \mathbb{R}^3 \setminus \{0\}$, which are pairwise orthogonal, i.e.

$$\forall i, j = 1, ..., 5 : i \neq j \implies \langle v_i, v_j \rangle = 0$$

Assume there exist five pairwise orthogonal, non-zero vectors $v_1, ..., v_5 \in \mathbb{R}^3$. In $\mathbb{R}^3$, there are at most three linearly independent vectors. Thus, the vectors are linearly dependent, which means

$$\exists a_i : \sum_{i=1}^5 a_i v_i = 0,$$

with at least one $a_i \neq 0$. Without loss of generality, assume that $a_1 = -1$, resulting in

$$v_1 = a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5$$

As the vectors are assumed to be pairwise orthogonal, we can derive

$$||v_1||^2 = \langle v_1, v_1 \rangle =$$

$$= \langle v_1, a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 \rangle =$$

$$= a_2 \langle v_1, v_2 \rangle + a_3 \langle v_1, v_3 \rangle + a_4 \langle v_1, v_4 \rangle + a_5 \langle v_1, v_5 \rangle =$$

$$= 0 + 0 + 0 + 0 = 0$$

$$\implies v_1 = 0,$$

which contradicts the assumption of pairwise orthogonal, non-zero vectors.

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1See e.g. https://en.wikipedia.org/wiki/Subgroup_test for a proof if this is not clear to you.