Convex Optimization for Machine Learning and Computer Vision

Lecture: Dr. Tao Wu
Exercises: Emanuel Laude, Zhenzhang Ye
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Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 4

Room: 02.09.023

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Convex cone (10+10 Points)

Exercise 1 (4 points)**.** Assume $J : \mathbb{E} \to \mathbb{R}$, prove following facts of convex conjugate:

- $\tilde{J}(\cdot) = \alpha J(\cdot) \Rightarrow \tilde{J}^*(\cdot) = \alpha J^*(\cdot/\alpha)$, $\alpha > 0$.
- $\tilde{J}(\cdot) = J(\cdot - z) \Rightarrow \tilde{J}^*(\cdot) = J^*(\cdot) + \langle \cdot, z \rangle$.

Solution.

- Using the definition of convex conjugate:

$$\begin{aligned}\tilde{J}(\cdot) &= \sup_u \langle u, \cdot \rangle - \tilde{J}(u) \\ &= \sup_u \langle u, \cdot \rangle - \alpha J(u) \\ &= \alpha \underbrace{\sup_u \langle u, \cdot/\alpha \rangle - J(u)}_{J^*(\cdot/\alpha)} \\ &= \alpha J^*(\cdot/\alpha)\end{aligned}$$

- $\tilde{J}(\cdot) = \sup_u \langle u, \cdot \rangle - J(u - z)$. Define $v = u - z$ and by substitution we have:

$$\begin{aligned}\tilde{J}(\cdot) &= \sup_v \langle v + z, \cdot \rangle + J(v) \\ &= \sup_v \langle v, \cdot \rangle + J(v) + \langle z, \cdot \rangle \\ &= J^*(\cdot) + \langle \cdot, z \rangle\end{aligned}$$

Exercise 2 (6 points)**.** Assume $J : \mathbb{R}^n \to \mathbb{R}$, compute the convex conjugate of following functions:

- $J(u) = \frac{1}{q}||u||_q^q = \sum_{i=1}^n \frac{1}{q}|u_i|^q$, $q \in [1, +\infty]$.
- $J(u) = \sum_{i=1}^n u_i \text{log} u_i + \delta_{\triangle^{n-1}}(u)$.
- $J(u) = \begin{cases} \frac{1}{2}\left\|u\right\|_2^2, & \left\|u\right\|_2 \leq \epsilon \\ +\infty, & \text{otherwise} \end{cases}$

Solution. • $J^*(v) = \sup_u \langle u, v\rangle - J(u)$. Since it is separable, we apply first-order optimality condition elementwisely:

$$\sup_{u_i} \langle u_i, v_i\rangle - \frac{1}{q}(|u_i|)^q \Rightarrow 0 = v_i - |u_i|^{q-1}\text{sign}(u_i) \Rightarrow u_i = |v_i|^{1/(q-1)}\text{sign}(v_i)$$

Substitute u_i back to the first equation, we have

$$\begin{aligned} J^*(v)_i &= |v_i|^{q/(q-1)} - \frac{1}{q}|v_i|^{q/(q-1)} \\ &= (1 - \frac{1}{q})|v_i|^{q/(q-1)} \\ &= (1 - \frac{1}{q})|v_i|^{1/(1-\frac{1}{q})} \end{aligned}$$

Substituting $\frac{1}{p} = 1 - \frac{1}{q}$, we get $J^*(v) = \frac{1}{p}||v||_p^p$.

- Consider the convex conjugate elementwisely: $J^*(v) = \sup_u \sum_i^n u_i v_i - u_i \text{log} u_i - \delta_{\triangle^{n-1}}(u)$. Let's consider the following minimization problem given v_i:

$$\begin{aligned} &\min_u \sum_i^n u_i \text{log} u_i - u_i v_i \\ &\text{s.t.} \mathbb{1} u = 1 \end{aligned}$$

where $\mathbb{1} = [1, \ldots, 1] \in \mathbb{R}^n$. It is obvious that this two problems share the same optimal variable u^* and the domain of log implies $u_i > 0$. Since the feasible set is compact and original energy function is continuous, the KKT condition holds on u^*. Therefore, we have certain $\lambda \in \mathbb{R}$ such that

$$\text{log} u_i^* + 1 - v_i + \lambda = 0, \ \forall i = 1, \ldots, n$$

which give $u_i^* = \exp\{-\lambda + v_i - 1\}$. Additionally, $\sum_{i=1}^n u_i^* = 1$. We can get

$$0 = \log(\sum_{i=1}^n \exp\{-\lambda + v_i - 1\}) = \log(\exp\{-\lambda - 1\} \sum_{i=1}^n e^{v_i}) = (-\lambda - 1) + \log(\sum_{i=1}^n e^{v_i})$$

Now, substitute u^* back into the convex conjugate and we can get

$$\begin{aligned} J(v)^* &= \sum_i^n \exp\{-\lambda + v_i - 1\} v_i - \exp\{-\lambda + v_i - 1\}(-\lambda + v_i - 1) \\ &= \sum_i^n -\exp\{-\lambda + v_i - 1\}(-\lambda - 1) \\ &= -(-\lambda - 1) = \log(\sum_{i=1}^n e^{v_i}) \end{aligned}$$

- Rewrite the convex conjugate as $J^*(v) = \sup_{\|u\|_2 \le \epsilon} \langle u, v\rangle - \frac{1}{2}\|u\|_2^2$. We first try to find the corresponding u^*.

$$
\begin{aligned}
u^* &= \text{argmin}_{\|u\|_2 \le \epsilon} \frac{1}{2}\|u\|_2^2 - \langle u, v\rangle + \frac{1}{2}\|v\|_2^2 \\
&= \text{argmin}_{\|u\|_2 \le \epsilon} \frac{1}{2}\|u - v\|_2^2
\end{aligned}
$$

which is a projection problem i.e. project v into a convex set $\{u : \|u\|_2 \le \epsilon\}$. Therefore, if $\|v\|_2 \le \epsilon$, $u^* = v$. Otherwise, $u^* = \epsilon\frac{v}{\|v\|}$.

$$
J^*(v) = \begin{cases} \frac{1}{2}\|v\|_2^2, & \|v\|_2 \le \epsilon \\ \epsilon\|v\|_2^2 - \frac{1}{2}\epsilon^2, & \text{otherwise} \end{cases}
$$

Exercise 3 (10 Points)**.**

Definition (Slater's condition)**.** Let $J : \mathbb{R}^n \to \mathbb{R}$, $G : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable and convex, and $H : \mathbb{R}^n \to \mathbb{R}^l$ be affine linear i.e. $Au + b = 0$. Let $U := \{u \in \mathbb{R}^n : g_i(u) \le 0, h_j(u) = 0,\ 1 \le i \le m, 1 \le j \le l\}$ denote the feasible set. The condition

$$
\exists\, u \in U \text{ s.t. } g_i(u) < 0, h_j(u) = 0, \forall\, 1 \le i \le m, 1 \le j \le l
$$

is called Slater's condition

Definition (Polar cone)**.** For a set C, the polar cone of C is defined as

$$
C^o = \{y \in \mathbb{E} : \langle y, d\rangle,\ \forall d \in C\}.
$$

Definition (Tangent cone)**.** Let $U \subset \mathbb{E}$ be convex and $u \in U$. Then the tangent cone $T_U(u)$ is defined as

$$
T_U(u) = \{d \in \mathbb{E} : \exists u_i \in U \text{ with } u_i \to u \text{ and } \exists t_i \to 0^+, \text{ s.t. } \lim_{i \to +\infty} \frac{u_i - u}{t_i} = d\}
$$

Now consider following constrainted optimization problem:

$$
\begin{aligned}
&\min_u\ J(u) \\
&s.t.\ g_i(u) \le 0, && i = 1, \dots, m \\
&\qquad h_j(u) = A_j u + b_j = 0, && j = 1, \dots, l
\end{aligned}
$$

where J and g_i are continuously differentiable and convex functions and h_j are affine linear. Let U be the feasible set defined as before and $U_1 := \{u \in \mathbb{R}^n : G(u) \le 0\}$ and $U_2 := \{u \in \mathbb{R}^n : H(u) = 0\}$. Assume Slater's condition holds in U.

1. Using following theorem:

   Theorem 1. Let $f_1, \dots, f_n$ are proper convex functions on $\mathbb{R}^n$, and let $f = f_1 + \cdots + f_m$. If the convex sets $\text{ri}(\text{dom} f_i)$, $i = 1, \dots, m$ have a point in common, then

   $$
   \partial f(u) = \partial f_1(u) + \cdots + \partial f_n(u),\ \forall u.
   $$

prove that $N_U(u) = N_{U_1}(u) + N_{U_2}(u)$ where $N_U(u)$ is the normal cone of U at u.

2. Prove that $N_{U_2}(u) = \{\sum_{j=1}^{l} \mu_j \nabla h_j(u) : \mu \in \mathbb{R}^l\}$.

3. Deduce that $T_{U_1}(u) = \{d \in \mathbb{E} : \nabla G_{\mathcal{A}}(u) d \leq 0\}$, where $\mathcal{A}(u) = \{i : g_i(u) = 0, i = 1, \ldots, m\}$ is called active set.

   Hint: Firstly, show that $\{d \in \mathbb{E} : \nabla G_{\mathcal{A}}(u) d \leq 0\} \subset \mathrm{cl}(\{d \in \mathbb{E} : \nabla G_{\mathcal{A}}(u) d < 0\}) \subset T_{U_1}(u)$. For the first "$\subset$" relation, consider the linear combination of a boundary point and an inner point. Then show $T_{U_1}(u) \subset \{d \in \mathbb{E} : \nabla G_{\mathcal{A}}(u) d \leq 0\}$.

4. Show that $N_{U_1}(u) = \{\sum_{i=1}^{m} \lambda_i \nabla g_i(u) : \lambda_i \geq 0, \lambda_i g_i(u) = 0, i = 1, \ldots, m\}$. You can use following two theorems:

   Theorem 2. If a set $C \subset \mathbb{E}$ is closed and convex, then the bipolar cone is itself i.e. $C^{oo} = C$.

   Theorem 3. Let $C \subset \mathbb{E}$ be a nonempty, convex set and let $u \in C$. Then the normal cone of C at u is the polar cone of the tangent cone of C at u. That is

$$N_c(u) = (T_c(u))^o.$$

5. Show that $u^* \in U$ satisfies that $-\nabla J(u^*) \in N_U(u^*)$ if and only if u^* is a minimizer.

Solution. 1. Firstly, let's define two indicator functions $f_1 = \delta_{U_1}(u)$ and $f_2 = \delta_{U_2}(u)$. It's clear that U_1 and U_2 are closed convex subset. As we know that the subdifferential of indicator function is the corresponding normal cone, we can get $\partial f_1(u) = N_{U_1}(u)$ and $\partial f_2(u) = N_{U_2}(u)$.
   Since the slater condition is satisfied in U, therefore, we can find a common point in $\mathrm{ri}(\mathrm{dom}(f_i))$, $i = 1, 2$. By applying the theorem, we finally get $N_U(u) = N_{U_1}(u) + N_{U_2}(u)$.

2. In fact, if we write the set into a matrix format, we can get $\{\sum_{j=1}^{l} \mu_j \nabla h_j(u) : \mu \in \mathbb{R}^l\} = \mathrm{ran}(\nabla H(u) = \mathrm{ran}(A^T)$. Recall the definition of normal cone:

$$N_{U_2}(u) = \{d \in \mathbb{E} : \langle d, v - u \rangle \leq 0, \forall v \in U_2\}.$$

   First, we show that $\mathrm{ran}(A^T) \subset \{d \in \mathbb{E} : \langle d, v - u \rangle \leq 0, \forall v \in U_2\}$. Pick $d = A^\top x$ for a certain x, $\langle d, v - u \rangle = \langle A^\top x, v - u \rangle = \langle x, A(v - u) \rangle = 0$
   Conversely, let's pick a d such that $\langle d, v - u \rangle \leq 0$. Since $(v - u) \in \ker(A)$ i.e. $A(v - u) = 0$, we have $\langle d, A(v - u) \rangle = 0, \forall v \in U_2$. This implies that d must be in the orthogonal plane of $\ker(A)$. So $d \in \ker(A)^\perp$ which is as same as $d \in \mathrm{ran}(A^T)$.

3. We first show that $\{d \in \mathbb{E} : \nabla G_{\mathcal{A}}(u)d < 0\} \subset T_{U_1}(u)$. Let d be such that $\nabla G_{\mathcal{A}}(u)d < 0$. Then for all sufficiently small $t > 0$, using Taylor's expansion we have

$$G_{\mathcal{A}}(u + td) = \underbrace{G_{\mathcal{A}}(u)}_{=0} + t\nabla G_{\mathcal{A}}(u)d + o(t) < 0.$$

We can always construct suffient small t to satisfy the definition of tangent cone. Since T_{U_2} is closed by definition, we get

$$\mathrm{cl}(\{d \in \mathbb{E} : \nabla G_{\mathcal{A}}(u)d < 0\}) \subset T_{U_1}(u).$$

According to the slater's condition, let's denote that $G(\bar{u}) < 0$ for a certain $\bar{u} \in U_1$. Consider a vector d with $\nabla G_{\mathcal{A}}(u)d \leq 0$. Using the property of gradient of convex function, we get for $\bar{d} := \bar{u} - u$

$$\begin{aligned}\nabla G_{\mathcal{A}}(u)\bar{d} &\leq \underbrace{G_{\mathcal{A}}(\bar{u})}_{<0} - \underbrace{G_{\mathcal{A}}(u)}_{=0} \\ \nabla G_{\mathcal{A}}(u)\bar{d} &< 0\end{aligned}$$

To show the left subset, we construct a linear combination of $\bar{d}$ and d with $0 < \lambda \leq 1$:

$$\nabla G_{\mathcal{A}}(u)(\lambda\bar{d} + (1 - \lambda)d) < 0,$$

which intuitionly means d is a boundary point. Therefore, $d \in \mathrm{cl}(\{d \in \mathbb{E} : \nabla G_{\mathcal{A}}(u)d < 0\})$.
Now consider the other direction. Pick a $d \in T_{U_1}(u)$. Therefore, we have a sequence $u_i \to u$ and $t_i \to 0^+$ such that

$$\lim_{i\to+\infty} \frac{u_i - u}{t_i} = d.$$

Rewrite the limitation, we have $u_i = u + t_i d$. Further, using the convexity of $G_{\mathcal{A}}(u)$:

$$\begin{aligned}0 &\geq G_{\mathcal{A}}(u + t_i d) \\ &\geq G_{\mathcal{A}}(u) + t_i\nabla G_{\mathcal{A}}(u)d \\ &= t_i\nabla G_{\mathcal{A}}(u)d.\end{aligned}$$

which shows $T_{U_1} \subset \{d \in \mathbb{E} : \nabla G_{\mathcal{A}}(u)d \leq 0\}$.

4. Denote $\tilde{N}_{U_1}(u) := \{\sum_{i=1}^m \lambda_i \nabla g_i(u) : \lambda_i \geq 0, \lambda_i g_i(u) = 0, i = 1, \ldots, m\}$. Rewrite it into following way:

$$\begin{aligned}\tilde{N}_{U_1}(u) &= \{\sum_{i=1}^m \lambda_i \nabla g_i(u) : \lambda_i \geq 0, \forall i = 1, \ldots, m, \lambda_i = 0, \forall i \notin \mathcal{A}(u)\} \\ &= \{\xi \in \mathbb{E} : \xi = (\nabla G_{\mathcal{A}}(u))^T\kappa, \kappa \geq 0\}.\end{aligned}$$

This set is clearly closed and $(\tilde{N}_{U_1}(u))^o = T_{U_1}(u)$. By applying the two theorems, we have

$$N_{U_1}(u) = (T_{U_1}(u))^o = (\tilde{N}_{U_1}(u))^{oo} = \tilde{N}_{U_1}(u).$$

5. Using previous results, we have following equation for a fixed $\lambda_i^*, i = 1, \dots, m$ and $\mu_j^*, j = 1, \dots, l$:

$$0 = \nabla J(u^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(u^*) + \sum_{j=1}^{l} \mu_j^* \nabla h_j(u^*)$$

Construct a new function $\mathcal{L}(u, \lambda, \mu) := J(u) + \sum_{i=1}^{m} \lambda_i g_i(u) + \sum_{j=1}^{l} \mu_j h_j(u)$. Above equation can be viewed as the first-order optimality condition. Therfore, for any $u \in U$, we can get:

$$\begin{aligned} J(u^*) &= \mathcal{L}(u^*, \lambda^*, \mu^*) \\ &\leq \mathcal{L}(u, \lambda^*, \mu^*) \\ &= J(u) + \underbrace{\sum_{i=1}^{m} \lambda_i^* g_i(u)}_{\leq 0} + \underbrace{\sum_{j=1}^{l} \mu_j^* h_j(u)}_{=0} \\ &\leq J(u) \end{aligned}$$

This proof holds for both directions.