

Weekly Exercises 2

Room: 02.09.023

Wed, 31.05.2017, 14:00-16:00

Submission deadline: Tue, 30.05.2017, 23:59 to laehner@in.tum.de

Mathematics: Manifolds

Exercise 1 (3 points). 1. Consider the function

$$\begin{aligned} g_1 : \mathbb{R} &\rightarrow \mathbb{R}^2 \\ t &\mapsto (t^3, t^2) \end{aligned} \tag{1}$$

Calculate the differential of g_1 .

2. Reason whether g_1 is an explicit representation of a manifold.

Tip: A vector is not full-rank if its 0.

3. Consider the next function

$$\begin{aligned} g_2 : \mathbb{R} &\rightarrow \mathbb{R}^3 \\ t &\mapsto (t, t^3, t^2) \end{aligned} \tag{2}$$

Reason whether g_2 is an explicit representation of a manifold.

Tip: Ex. 5 can help imagining g_1, g_2 .

Solution. 1. $D_{g_1} = (3t^2, 2t)$

2. For $t = 0 : D_{g_1} = 0$ and therefore g_1 does not describe a manifold.

3. $D_{g_2} = (1, 3t^2, 2t)$

$D_{g_2} \neq 0 \forall t \in \mathbb{R}$ and therefore g_2 describes a manifold.

Exercise 2 (3 points). Consider the set $O(n)$ including orthogonal matrices in $\mathbb{R}^{n \times n}$

$$O(n) = \{A \in \mathbb{R}^{n \times n} \mid AA^\top = Id\}$$

and the following map

$$\begin{aligned} \varphi : \mathbb{R}^{n \times n} &\rightarrow Sym(n) \\ A &\mapsto AA^\top \end{aligned}$$

1. Calculate the differential of φ .

2. $O(n)$ can be described by the implicit formulation $O(n) = \varphi^{-1}(Id)$. Proof that $O(n)$ is a manifold by showing that the differential of φ is of full rank.
3. What is the dimension of $O(n)$? Explain your answer.

Solution. 1. The differential is the best linear approximation of φ .

$$\begin{aligned}
 D_\varphi[A](H) &= \varphi(A + H) - \varphi(A) \\
 &= (A + H)(A + H)^\top - AA^\top \\
 &= AA^\top + HA^\top + AH^\top + HH^\top \\
 &= HA^\top + AH^\top + \mathcal{O}(H^2)
 \end{aligned}$$

2. Show that for every $A \in O(n), B \in Sym(n)$ exists a $H \in \mathbb{R}^{n \times n}$ such that $D_\varphi[A](H) = B$. Choose $H = \frac{1}{2}BA$ then

$$\begin{aligned}
 D_\varphi[A](H) &= HA^\top + AH^\top \\
 &= \frac{1}{2}BAA^\top + A\left(\frac{1}{2}BA\right)^\top \\
 &= \frac{1}{2}B + \frac{1}{2}AA^\top B^\top \\
 &= \frac{1}{2}(B + B^\top) = B
 \end{aligned}$$

Fun Fact: $\varphi^{-1}(X)$ is a manifold as long as X is invertible. (Otherwise you can not find a proper H .)

3.

$$\begin{aligned}
 \dim(O(n)) &= \dim(\text{domain}(\varphi)) - \text{co-dimension} \\
 &= \dim(\mathbb{R}^{n \times n}) - \dim(Sym(n)) \\
 &= n^2 - \frac{n \cdot (n + 1)}{2} = \frac{n \cdot (n - 1)}{2}
 \end{aligned}$$

Exercise 3 (1 point). Find a coordinate map $\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ of the torus

$$T = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \left(\sqrt{x^2 + y^2} - a \right)^2 + z^2 = r^2 \right\}$$

with $a > r > 0$. (It can of course not cover the complete manifold.)

Solution. Since in the defining equation of T only the distance $\sqrt{x^2 + y^2}$ in the xy -plane from the origin plays a role, the set T is symmetric under rotations around the z -axis. It is therefore sufficient to understand the set in the xz -plane - the hole torus then shows up via rotation around the z -axis. For $y = 0$ we get

$$(|x| - a)^2 + z^2 = r^2.$$

Those are two circles with radius r with centers $(x = a, z = 0)$ and $(x = -a, z = 0)$. A parametrization can now be given via a rotation of one of them around the z -axis:

$$\mathbf{x} : (0, 2\pi)^2 \rightarrow \mathbb{R}^3, \quad (u, v) \mapsto \begin{pmatrix} \cos(v)(a + r \cos(u)) \\ \sin(v)(a + r \cos(u)) \\ r \sin(u) \end{pmatrix}$$

The coordinate map is smooth since $Dx(u)$ is of full rank:

$$Dx(u) = \begin{pmatrix} -r \cos(v) \sin(u) & -\sin(v)(a + r \cos(u)) \\ -r \sin(v) \sin(u) & \cos(v)(a + r \cos(u)) \\ r \cos(u) & 0 \end{pmatrix}$$

Exercise 4 (1 point). The circle with radius r can be represented by the implicit formulation

$$\begin{aligned} \varphi : \mathbb{R}^2 &\rightarrow \mathbb{R} \\ x &\mapsto \sqrt{\langle x, x \rangle} \\ C_r &= \varphi^{-1}(r) \end{aligned}$$

Calculate the curvature on each point of C_r .

Solution.

$$\begin{aligned} \nabla \varphi &= \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \\ \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \end{pmatrix} \\ \frac{\nabla \varphi}{\|\nabla \varphi\|} &= \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \\ \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \end{pmatrix} \\ \operatorname{div} \left(\frac{\nabla \varphi}{\|\nabla \varphi\|} \right) &= \frac{x_1^2 + x_2^2}{(x_1^2 + x_2^2)^{3/2}} \\ &= \frac{r^2}{r^3} = \frac{1}{r} \end{aligned}$$

Programming: Working with Matlab

You can use MATLAB in the computer labs or install it on your own computer (licences are free for TUM students, see matlab.rbg.tum.de). If you have never used MATLAB before, going through a tutorial before doing the exercises will probably help (there is an interactive MATLAB Academy).

Exercise 5 (1 point). Consider the functions (1), (2) from Ex. 1. Plot both functions for $t \in [-2, 2]$ in two subplots. Include a figure of your result in your submission. Tip: Looking up `subplot`, `plot`, `plot3` and function handles (if you feel fancy) might be helpful.

Solution.

Figure 1: Possible solution for Ex. 5.

Exercise 6 (3 points). The unit sphere can be described as

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

Consider the three following coordinate maps: ($C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$)

$$\begin{aligned} x_1 : C &\rightarrow \mathbb{R}^3 \\ (u, v) &\mapsto (u, v, \sqrt{1 - u^2 - v^2}) \end{aligned} \tag{3}$$

$$\begin{aligned} x_2 : C &\rightarrow \mathbb{R}^3 \\ (u, v) &\mapsto (u, v, -\sqrt{1 - u^2 - v^2}) \end{aligned} \tag{4}$$

$$\begin{aligned} x_3 :] - 10, 10[\times] - 10, 10[&\rightarrow \mathbb{R}^3 \\ (u, v) &\mapsto \frac{1}{u^2 + v^2 + 1} (2u, 2v, u^2 + v^2 - 1) \end{aligned} \tag{5}$$

$$\begin{aligned} x_4 :]0, 1[\times]0, 2\pi[&\rightarrow \mathbb{R}^3 \\ (h, \theta) &\mapsto (\sin(h\pi) \cos(\theta), \sin(h\pi) \sin(\theta), \cos(h\pi)) \end{aligned} \tag{6}$$

The height function h on S is defined as follows:

$$\begin{aligned} h : S &\rightarrow \mathbb{R} \\ (x, y, z) &\mapsto z \end{aligned} \tag{7}$$

1. Make a figure plotting the images of x_1, x_2, x_3, x_4 as surfaces with h as a function on the surface.
2. Plot $(h \circ x_i), i \in \{1, \dots, 4\}$ as 2D images.
3. Which pairs of coordinates maps together properly define the unit sphere as a manifold?

Tip: Look up the Matlab functions `ndgrid`, `surf` (set z-coordinate to NaN if you do not want some values to be plotted, `surf(..., 'EdgeAlpha', 0)` will make the edges invisible), `imagesc` and logical indexing.

Figure 2: Possible solution for Ex. 6.

Solution.