Analysis of Three-Dimensional Shapes
F. R. Schmidt, M. Vestner, Z. Lähner
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Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 8

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Mathematics: Stiffness matrix

Let X be a vector space. An *inner product* is a function $f : X \times X \to \mathbb{C}$ with the following properties:

1. $f(x, x) \geq 0 \quad \forall\, x \in X$ and $f(x, x) = 0 \Leftrightarrow x = 0$
2. $f(x, y) = \overline{f(y, x)}$
3. $f(x + \alpha x', y) = f(x, y) + \alpha f(x', y) \quad \forall\, x, x', y \in X, \alpha \in \mathbb{C}$

The standard inner product on $X = \mathbb{C}$ is defined as $\langle x, y\rangle = x^\top \bar{y}$. If $M \in \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix the M-inner product is defined by $\langle x, y\rangle_M = x^\top M \bar{y}$.

A linear operator $T : X \to X$ is called *self-adjoint* w.r.t. an inner product f if the following holds:

$$f(Tx, y) = f(x, Ty)$$

An *eigenvector* is an element $0 \neq x \in X$ for which there exists a scalar $\lambda \in \mathbb{C}$ such that

$$Tx = \lambda x$$

The scalar λ is called *eigenvalue*.

Exercise 1. Let $L = M^{-1}S \in \mathbb{R}^{n\times n}$ be self-adjoint w.r.t. the M inner product. Show that the following statements hold.

1. S is symmetric (self-adjoint) w.r.t. to the standard inner product.
2. The eigenvalues of L are real.
3. The eigenvectors v_i, v_j with respective eigenvalues $\lambda_i \neq \lambda_j$ are orthogonal.
4. $v_1, ..., v_k$ are eigenvectors of L with the same eigenvalue λ, then $\sum_i \alpha_i v_i$ is also an eigenvector with eigenvalue λ.

Exercise 2. Show that the diagonal entries $\mathbf{S}_{ii} = \int_{\mathcal{M}} \langle \nabla \psi_i, \nabla \psi_i \rangle$ of the stiffness matrix satisfy:

$$\mathbf{S}_{ii} = \sum_{(i,j) \text{ edge at } i} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} = -\sum_j \mathbf{S}_{ij}$$

Hints

- Consider each traingle independently
- $\int_T 1 dp = \int_{T_{ref}} \sqrt{\det g} du = \text{area}(T)$.
- The area of a triangle can be calculated as the half of the product of an edge length and the corresponding height of the triangle.

Programming: Stiffness matrix

Exercise 3. Download the supplementary material from the homepage. It contains four files describing two 3D triangular meshes.

1. Implement a function `stiffness_matrix.m` that takes a triangle mesh and returns a `sparse` stiffness matrix.
2. Use the `eigs` command to get the first four (ordered by magnitude of the eigenvalue, from small to big) solutions of the generalized eigenvalue problem

$$\lambda \mathbf{M} \phi_i = -\mathbf{S} \phi_i$$

   and visualize them

   - as color coded functions on the shapes.
   - as embeddings of the shapes in $\mathbb{R}^3$ (do not use the first one. Why?).

Hints

- Recap the construction of the mass matrix from sheet 4.
- For each triangle calculate the cot of all three angles and add them to the corresponding positions in the stiffness matrix.
- the `sparse` command automatically adds values if an entry is assigned multiple times.