

Analysis of 3D Shapes (IN2238)

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12. Second Fundamental Form

Covariant Derivative

Christoffel Symbols

Given a coordinate map $x: U \rightarrow M$ of the n -dimensional manifold $M \subset \mathbb{R}^{n+1}$, the (intrinsic) Riemannian metric is given as

$$g: U \rightarrow \mathbb{R}^{n \times n} \quad g_{ij}(u) = \langle \partial_i x(u), \partial_j x(u) \rangle$$

While the first derivatives $\partial_i x(u)$ lie in the n -dimensional vector space $T_{x(u)}M$, the second derivatives might contain a normal component, i.e.,

$$\partial_{ij} x(u) = \sum_{k=1}^n \Gamma_{ij}^k(u) \partial_k x(u) + \alpha_{ij}(u) N(u)$$

The n^3 scalar functions $\Gamma_{ij}^k: U \rightarrow \mathbb{R}$ are called **Christoffel symbols**. They are symmetric in i and j , i.e., $\Gamma_{ij}^k = \Gamma_{ji}^k$. (Why?)

Christoffel Symbols and Metric

Using $\partial_i g_{j\ell}(u) = \langle \partial_i \partial_j x(u), \partial_\ell x(u) \rangle + \langle \partial_i \partial_\ell x(u), \partial_j x(u) \rangle$, we obtain

$$\begin{aligned} \tilde{\Gamma}_{\ell ij}(u) &:= \frac{1}{2} [\partial_i g_{j\ell}(u) + \partial_j g_{\ell i}(u) - \partial_\ell g_{ij}(u)] \\ &= \frac{1}{2} [\langle \partial_i \partial_j x(u), \partial_\ell x(u) \rangle + \langle \partial_i \partial_\ell x(u), \partial_j x(u) \rangle + \langle \partial_j \partial_\ell x(u), \partial_i x(u) \rangle + \\ &\quad \langle \partial_j \partial_i x(u), \partial_\ell x(u) \rangle - \langle \partial_i \partial_\ell x(u), \partial_j x(u) \rangle - \langle \partial_j \partial_i x(u), \partial_\ell x(u) \rangle] \\ &= \langle \partial_{ij} x(u), \partial_\ell x(u) \rangle = \sum_{k=1}^n \Gamma_{ij}^k(u) g_{k\ell}(u) \end{aligned}$$

If we use the notation $g^{ij}(u) := (g(u)^{-1})_{ij}$, we obtain

$$\sum_{\ell=1}^n g^{k\ell}(u) \tilde{\Gamma}_{\ell ij}(u) = \sum_{k=1}^n \sum_{\ell=1}^n g^{k\ell}(u) g_{\ell k'}(u) \Gamma_{ij}^{k'}(u) = \Gamma_{ij}^k(u)$$

Christoffel Symbols are Intrinsic

In summary, we have

$$\partial_{ij} x = \sum_{k=1}^n \Gamma_{ij}^k \partial_k x + \alpha_{ij} N$$

with the **intrinsic** Christoffel symbols

$$\Gamma_{ij}^k = \sum_{\ell=1}^n \frac{1}{2} g^{k\ell} [\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}]$$

The expression $\sum_{k=1}^n \Gamma_{ij}^k \partial_k x$ can also be seen as an intrinsic derivative of the vector field $\partial_j x$ in the direction of $\partial_i x$.

This derivative is called **covariant derivative** $\nabla_{\partial_i x} \partial_j x$.

Example: Sphere

Given the coordinate map

$$\begin{aligned} x: \left] -\frac{\pi}{3}, \frac{\pi}{3} \right[\times \left] -\frac{\pi}{3}, \frac{\pi}{3} \right[&\rightarrow \mathbb{S}^2 \\ (\alpha_1, \alpha_2) &\mapsto \begin{pmatrix} \cos(\alpha_1) \cos(\alpha_2) \\ \sin(\alpha_1) \cos(\alpha_2) \\ \sin(\alpha_2) \end{pmatrix} \end{aligned}$$

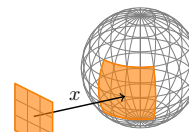
we obtain the Riemannian metric

$$g(\alpha_1, \alpha_2) = \begin{pmatrix} \cos(\alpha_2)^2 & 0 \\ 0 & 1 \end{pmatrix}$$

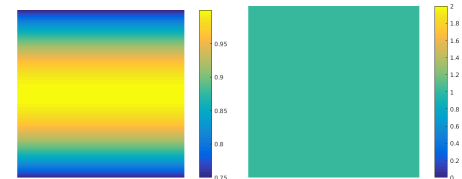
and the Christoffel symbols

$$\Gamma^1(\alpha_1, \alpha_2) = -\frac{\sin(2\alpha_2)}{2 \cos(\alpha_2)^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Gamma^2(\alpha_1, \alpha_2) = \frac{\sin(2\alpha_2)}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Example: Christoffel Symbols

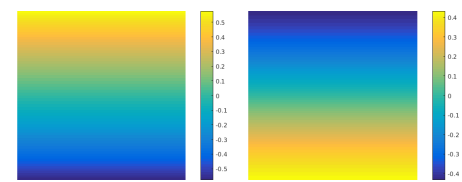


Parametrization



g_{11}

g_{22}

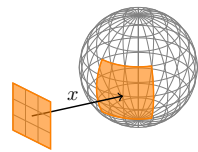


$\Gamma^1_{12} = \Gamma^1_{21}$

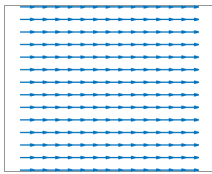
Γ^2_{11}

Example: Covariant Derivative

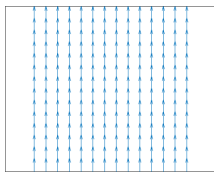
Covariant Derivative Geodesics Second Fundamental Form



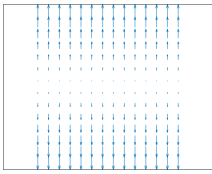
Parametrization



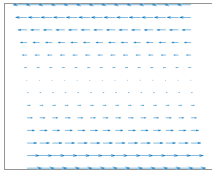
"∂1x"



"∂2x"



"∇∂1x∂1x"



"∇∂1x∂2x" = "∇∂2x∂1x"



"∇∂2x∂2x"

Covariant Derivative

Covariant Derivative Geodesics Second Fundamental Form

Given a coordinate map $x: U \rightarrow M$ of the n -dimensional manifold $M \subset \mathbb{R}^{n+1}$, and two vector fields Y and Z represented as ($p = x(u)$)

$$Y(p) = \sum_{i=1}^n y_i(u) \partial_i x(u) \quad Z(p) = \sum_{j=1}^n z_j(u) \partial_j x(u),$$

the **covariant derivative** $\nabla_Z Y$ is a vector field that can be represented as

$$[\nabla_{\partial_j x} \partial_i x](p) = \sum_{k=1}^n \Gamma_{ij}^k(u) \partial_k x(u)$$

$$[\nabla_{\partial_j x} Y](p) = \sum_{i=1}^n y_i(u) \nabla_{\partial_j x} \partial_i x(p) + \partial_j y_i(u) \partial_i x(u) \quad (\text{product rule in } Y)$$

$$[\nabla_Z Y](p) = \sum_{j=1}^n z_j(u) \cdot \nabla_{\partial_j x} Y(p) \quad (\text{linearity in } Z)$$

Extrinsic Formulation

Covariant Derivative Geodesics Second Fundamental Form

$\nabla_Z Y$ can be formulated in a simpler manner if Y and Z can be extended to the ambient space \mathbb{R}^{n+1} of M . To this end let

$$\tilde{Y}, \tilde{Z}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$$

with $\tilde{Y}|_M = Y$ and $\tilde{Z}|_M = Z$.

Then, we have for every $p \in M$

$$\nabla_Z Y(p) = \pi_{T_p M} (D\tilde{Y}(p) \cdot \tilde{Z}(p)),$$

where

$$\pi_{T_p M}: \mathbb{R}^{n+1} \rightarrow T_p M$$

is the orthogonal projection of the ambient space \mathbb{R}^{n+1} onto $T_p M$.

Geodesics

Shortest Path in Local Coordinates

Covariant Derivative Geodesics Second Fundamental Form

Given a coordinate map $x: U \rightarrow M$ of the n -dimensional manifold M , we like to find the shortest path $\gamma: [0, 1] \rightarrow U$ that connects two points $u_0, u_1 \in U$.

The length of γ is induced by the Riemannian metric $g: U \rightarrow \mathbb{R}^{n \times n}$ via

$$\text{length}(\gamma) = \int_0^1 \langle \dot{\gamma}(t), g(\gamma(t)) \cdot \dot{\gamma}(t) \rangle^{\frac{1}{2}} dt$$

It is often easier to consider the following energy function instead

$$E(\gamma) = \left[\int_0^1 \langle \dot{\gamma}(t), g(\gamma(t)) \cdot \dot{\gamma}(t) \rangle dt \right]^{\frac{1}{2}}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\text{length}(\gamma) \leq E(\gamma)$$

with equality iff $\|\dot{\gamma}\|_g \equiv \text{const}$, i.e., γ is uniformly parametrized.

Geodesics

Covariant Derivative Geodesics Second Fundamental Form

Let us select the two minimizers $\gamma^* \in \text{argmin length}(\cdot)$ and $\hat{\gamma} \in \text{argmin } E(\cdot)$. Further we assume that $\tilde{\gamma}^*$ is a uniform re-parametrization of γ^* .

Then we have

$$\text{length}(\gamma^*) = \text{length}(\tilde{\gamma}^*) = E(\tilde{\gamma}^*) \geq E(\hat{\gamma}) \geq \text{length}(\hat{\gamma}) \geq \text{length}(\gamma^*).$$

Therefore, we know

Every minimizer of E minimizes length [length($\hat{\gamma}$) = length(γ^*)]

The minimum of E is the minimal length [length($\hat{\gamma}$) = $E(\hat{\gamma})$]

The minimizer of E is uniformly parametrized [length($\hat{\gamma}$) = $E(\hat{\gamma})$]

Minimizing E provides us with a uniformly parametrized shortest path between two points. Every local minimum of E is called **geodesic**.

Geodesic Equation

Covariant Derivative Geodesics Second Fundamental Form

Given two points $u_0, u_1 \in U$, a geodesic $\gamma = (\gamma_1, \dots, \gamma_n): [0, 1] \rightarrow U$ that connects these points minimizes

$$E(\gamma_1, \dots, \gamma_n) := \int_0^1 \sum_{i,j=1}^n g_{ij}(\gamma(t)) \cdot \dot{\gamma}^i(t) \dot{\gamma}^j(t) dt$$

The **Euler-Lagrange equation** is

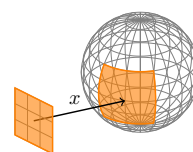
$$0 = \frac{\partial E}{\partial \gamma^k} = \sum_{i,j=1}^n \partial_k g_{ij}(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t) - \frac{d}{dt} \left[2 \sum_{i=1}^n g_{ik}(\gamma(t)) \dot{\gamma}^i(t) \right]$$

$$\ddot{\gamma}^k = - \langle \dot{\gamma}, \Gamma^k \dot{\gamma} \rangle$$

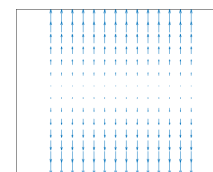
and can therefore be presented with respect to the Christoffel symbols.

Example: Geodesics

Covariant Derivative Geodesics Second Fundamental Form



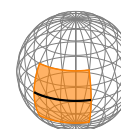
Parametrization



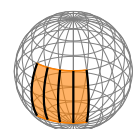
"∇∂1x∂1x"



"∇∂2x∂2x"



Equator



Meridians

Covariant Derivative along Curves

Covariant Derivative Geodesics Second Fundamental Form

Given a curve $\gamma: (0, L) \rightarrow U$ and a vector field X along the curve $c = x \circ \gamma$, we would like to define

$$\frac{\nabla}{dt} X := \nabla_{\dot{c}} X$$

To this end let $Y = \sum_{i=1}^n y^i \partial_i x$ be a vector field on M that coincides along c with X . Further let $Z = \sum_{i=1}^n z^i \partial_i x$ be a vector field that coincides along c with \dot{c} . Then we have $(p = x(u) = c(\tau))$

$$\nabla_Z Y(p) = \sum_{k=1}^n \left[\frac{d}{dt} (y^k \circ \gamma(t)) \Big|_{t=\tau} + \sum_{i,j=1}^n y^i(u) \Gamma_{ij}^k(u) z^j(u) \right] \partial_k x(u)$$

If we restrict this vector field to a vector field along the curve, it only depends on X and c , but not on the extension of Y and Z . Thus, $\frac{\nabla}{dt}$ is well defined.

Geodesic Equation of the Covariant Derivative

Covariant Derivative Geodesics Second Fundamental Form

Given a geodesic $c: (0, 1) \rightarrow M$, we have for $\dot{c} = \sum_{i=1}^n \dot{\gamma}^i \partial_i x$

$$\begin{aligned} \frac{\nabla}{dt} \dot{c} &= \sum_{k=1}^n \left[\dot{\gamma}^k + \sum_{i,j=1}^n \dot{\gamma}^i \Gamma_{ij}^k \dot{\gamma}^j \right] \partial_k x \\ &= \sum_{k=1}^n \left[\dot{\gamma}^k + \langle \dot{\gamma}, \Gamma^k \dot{\gamma} \rangle \right] \partial_k x = 0 \end{aligned}$$

The geodesic equation can therefore be written as

$$\frac{\nabla}{dt} \dot{c} = 0$$

Since $\frac{\nabla}{dt} \dot{c}$ measures how different a curve c is from a geodesic we can use it to define the **geodesic curvature** of a curve.

Geodesic curvature

Covariant Derivative Geodesics Second Fundamental Form

Given a curve $c: (0, L) \rightarrow \mathbb{R}^2$ parametrized by arc-length ($\|\dot{c}\| \equiv 1$), the curvature $\kappa(t)$ at $c(t)$ can be computed via

$$\kappa(t) = \frac{\det(\dot{c}(t), \ddot{c}(t))}{\|\dot{c}(t)\|^3} = \det(\dot{c}(t), \ddot{c}(t))$$

Given a curve $c: (0, L) \rightarrow M$ in the 2D manifold M that is parametrized by arc-length, we can compute the **geodesic curvature** $\kappa_g(t)$ by replacing \ddot{c} with $\frac{\nabla}{dt} \dot{c}$ and obtain

$$\kappa_g(t) = \det \left(\dot{c}(t), \frac{\nabla}{dt} \dot{c}(t) \right)$$

The geodesic curvature is 0 for geodesics and can therefore be understood as an intrinsic reformulation of the classical curvature of curves.

Second Fundamental Form

Covariant Derivative Geodesics Second Fundamental Form

Gauss Map

Covariant Derivative Geodesics Second Fundamental Form

Given a 2D manifold $M \subset \mathbb{R}^3$, we call a smooth mapping

$$N: M \rightarrow \mathbb{S}^2 \quad \forall p \in M: N(p) \perp T_p M$$

its **Gauss map**. For every 3D shape there exists such a mapping. (Why?)

If $x: U \rightarrow M$ is a coordinate mapping, we can always define a local Gauss map via

$$N: M \rightarrow \mathbb{S}^2$$

$$p \mapsto \frac{\partial_1 x(u) \times \partial_2 x(u)}{\|\partial_1 x(u) \times \partial_2 x(u)\|} \quad \text{for } u = x^{-1}(p)$$

If $M = f^{-1}(c)$ is given implicitly via a function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$, the Gauss map is given via $N(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|}$.

Shape Operator

Covariant Derivative Geodesics Second Fundamental Form

Given a 2D manifold $M \subset \mathbb{R}^3$ together with its Gauss map $N: M \rightarrow \mathbb{S}^2$, we call its differential the **shape operator** or **Weingarten mapping** S

$$S_p: T_p M \rightarrow T_{N(p)} \mathbb{S}^2$$

$$v \mapsto DN(p)[v]$$

Since $T_{N(p)} \mathbb{S}^2 = N(p)^\perp = T_p M$, $S_p: T_p M \rightarrow T_p M$ is an endomorphism.

If we choose a base of $T_p M$, we would obtain a 2×2 matrix, but this matrix would depend on the chosen base. Nonetheless, the eigenvalues of these matrices would remain the same.

The goal is to show that S_p can be put in diagonal form and that both eigenvalues are real.

Self-Adjointness of the Shape Operator

Covariant Derivative Geodesics Second Fundamental Form

We know (Linear Algebra) that self-adjoint endomorphisms are diagonalizable with real eigenvalues. Therefore, we have to prove that

$$\langle v_1, S_p(v_2) \rangle = \langle S_p(v_1), v_2 \rangle \quad \text{for all } v_1, v_2 \in T_p M$$

If v_1 and v_2 are co-linear this is obvious. If they are not co-linear, one can find a local coordinate map $x: U \rightarrow M$ with $x(0) = p$ and $v_i = \partial_i x(0)$.

Using $\langle N \circ x(u), \partial_i x(u) \rangle \equiv 0$ leads to

$$0 = \partial_1 \langle N \circ x(u), \partial_2 x(u) \rangle \Big|_{u=0} = \langle S_p(v_1), v_2 \rangle + \langle N(p), \partial_{12} x(0) \rangle$$

$$0 = \partial_2 \langle N \circ x(u), \partial_1 x(u) \rangle \Big|_{u=0} = \langle S_p(v_2), v_1 \rangle + \langle N(p), \partial_{21} x(0) \rangle$$

which proves the self-adjointness of the shape operator.

Principal Curvatures

Covariant Derivative Geodesics Second Fundamental Form

The two eigenvalues $\kappa_1(p)$ and $\kappa_2(p)$ of S_p are called **principal curvatures** and corresponding eigenvectors $v_1(p)$ and $v_2(p)$ are called **principal curvature directions**.

Note that $\kappa_g(p)$ along the geodesic c_i corresponding to $v_i(p)$ is 0 and the curvature of this curve coincides with $\kappa_i(p)$. In that sense, we can think of the principal curvatures as natural generalizations of the planar curvature.

We can derive two other curvatures from the principal curvatures:

$$H(p) := \frac{\kappa_1(p) + \kappa_2(p)}{2} = \frac{1}{2} \text{tr}(\mathcal{M}) \quad (\text{mean curvature})$$

$$K(p) := \kappa_1(p) \cdot \kappa_2(p) = \det(\mathcal{M}) \quad (\text{Gauss curvature})$$

given any representing matrix \mathcal{M} of S_p .

Given the shape operator $S_p: T_pM \rightarrow T_pM$, we can define the **Second Fundamental Form**

$$\mathbb{I}: T_pM \times T_pM \rightarrow \mathbb{R} \quad (v_1, v_2) \mapsto \langle S_p v_1, v_2 \rangle$$

This means, we have

$$\partial_{ij}x = \sum_{k=1}^n \Gamma_{ij}^k \partial_k x - \mathbb{I}(\partial_i x, \partial_j x) \cdot N$$

and the second fundamental form can be computed via

$$\mathbb{I}(\partial_i x, \partial_j x) = -\langle \partial_{ij}x, N \rangle.$$

Any coordinate map $x: U \rightarrow M$ provides for a base $\{\partial_1 x(u), \dots, \partial_n x(u)\}$ of T_pM for $p = x(u)$. In this base, the shape operator S_p can be written as

$$S_p(\partial_j x(u)) = \sum_{i=1}^n \nu_j^i \partial_i x(u)$$

This means, we have

$$\mathbb{I}(\partial_j x, \partial_k x) = \langle S_p(\partial_j x), \partial_k x \rangle = \sum_{i=1}^n \langle \nu_j^i \partial_i x, \partial_k x \rangle = \sum_{i=1}^n g_{ki} \nu_j^i$$

In other words the representing matrix ν of S_p satisfies the **Weingarten equations**

$$\nu = \mathcal{M}_{D_x}^x(S_p) = g^{-1} \cdot \mathbb{I}$$