

# Analysis of 3D Shapes (IN2238)

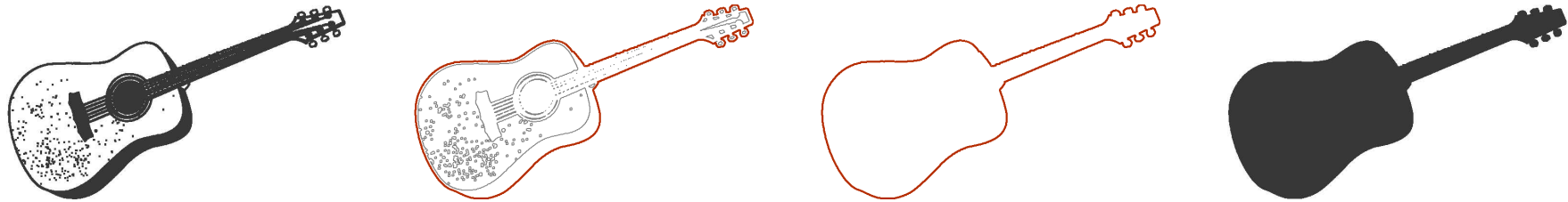
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## 2D Objects



To summarize the last lectures, we assume that a 2D object is an open set  $O \subset \mathbb{R}^2$  such that  $C = \partial O$  is a closed contour that is parameterized via

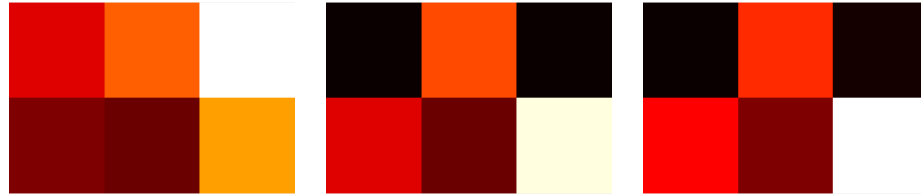
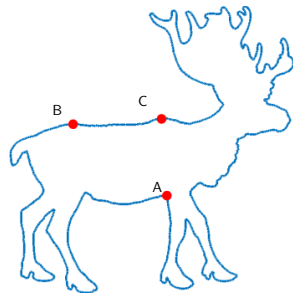
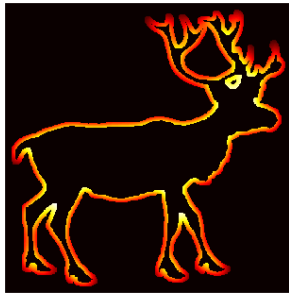
$$c: \mathbb{S}^1 \rightarrow \mathbb{R}^2.$$

If we choose a diffeomorphism  $m: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  defined on the parametrization domain  $\mathbb{S}^1$ , we obtain a different parametrization

$$c \circ m: \mathbb{S}^1 \rightarrow \mathbb{R}^2.$$

of the contour  $C = \partial O$ . Thus,  $c: \mathbb{S}^1 \rightarrow \mathbb{R}^2$  is **not a unique** representation.

## 2D Shapes



We also assume that we have a pointwise feature representation  $f: \mathbb{S}^1 \rightarrow \mathbb{R}^k$  such that shape-equivalent curves  $c_1, c_2: \mathbb{S}^1 \rightarrow \mathbb{R}^2$  lead to the same feature representation  $f_1 \equiv f_2$ .

The problem of shape matching can now be formulated as finding a mapping  $m: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that

$$f_1(s) \approx f_2(m(s)) \quad \text{for all } s \in \mathbb{S}^1$$

## 2D Shape Distance (naïve version)

Solving the shape matching problem results in minimizing the following energy

$$E_0(m) = \int_{\mathbb{S}^1} \text{dist}_{\mathcal{F}}(f_1(s), f_2 \circ m(s)) ds \quad m: \mathbb{S}^1 \rightarrow \mathbb{S}^1,$$

where  $\text{dist}_{\mathcal{F}}(\cdot, \cdot)$  measures the similarity of two features in the  $k$ -dimensional feature space  $\mathbb{R}^k$ .

Since  $E_0(m) \geq 0$  for all  $m$ , we can define for two curves  $c_1$  and  $c_2$  their “distance” as

$$\text{dist}_0(c_1, c_2) = \min_{m: \mathbb{S}^1 \rightarrow \mathbb{S}^1} \int_{\mathbb{S}^1} \text{dist}_{\mathcal{F}}(f_1(s), f_2 \circ m(s)) ds,$$

where  $f_i$  are the feature representation of  $c_i$  for  $i = 1, 2$ .

## Metric and Semi-Metric

We would like to use  $\text{dist}_0$  as a distance function for shapes. Nonetheless, we need some extra work in order to obtain a meaningful shape distance. To this end, we need to differentiate between a metric and a semi-metric.

**Definition 1.** Given a space  $X$ , we call  $d: X \times X \rightarrow \mathbb{R}_0^+$  a **metric** and  $X$  a **metric space** if

$$\begin{aligned} d(x, y) &= 0 && \Leftrightarrow && x = y && \text{(Positive Definiteness)} \\ d(x, y) &= d(y, x) && && && \text{(Symmetry)} \\ d(x, z) &\leq d(x, y) + d(y, z) && && && \text{(Triangle Inequality)} \end{aligned}$$

If  $d$  is only positive definite and symmetric, but does not necessarily satisfy the triangle inequality, we call  $d$  a **semi-metric**.

### Some Properties of $\text{dist}_0$

The mapping  $\text{dist}_0$  has the following properties for all object curves  $c_1, c_2: \mathbb{S}^1 \rightarrow \mathbb{R}^2$

$$\begin{array}{ll} \text{dist}_0(c_1, c_2) = 0 & \text{for } c_1 \sim c_2 \\ \text{dist}_0(c_1, c_1 \circ m) = 0 & \text{for all bijective } m: \mathbb{S}^1 \rightarrow \mathbb{S}^1 \end{array}$$

Nonetheless, the symmetry

$$\text{dist}_0(c_1, c_2) = \text{dist}_0(c_2, c_1)$$

is only possible if we restrict matchings  $m: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  to bijective functions.

Then we expect that given the optimal matching  $m$  between  $c_1$  and  $c_2$  would lead to the optimal matching  $m^{-1}$  between  $c_2$  and  $c_1$ .

## Linear Assignment

Looking for bijections  $m: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  lead to the LAP, which only considers permutations as a valid matching.

This means, we have with  $s_k = \exp\left(\frac{2\pi k}{N}i\right)$

$$\begin{aligned} E_0(m) &= \int_{\mathbb{S}^1} \text{dist}_{\mathcal{F}}(f_1(s), f_2 \circ m(s)) ds \\ &\approx \sum_{k=1}^N \text{dist}_{\mathcal{F}}[f_1(s_k), f_2 \circ m(s_k)] \cdot \frac{2\pi}{N} \\ E_0(m^{-1}) &= \int_{\mathbb{S}^1} \text{dist}_{\mathcal{F}}(f_2(s), f_1 \circ m^{-1}(s)) ds \\ &\approx \sum_{k=1}^N \text{dist}_{\mathcal{F}}[f_1 \circ m^{-1}(s_k), f_2(s_k)] \cdot \frac{2\pi}{N} \end{aligned}$$

## Restrictions of LAP

The LAP leads to a symmetric distance between object curves  $c_1$  and  $c_2$ , if the features are sampled equidistantly with respect to the chosen parametrization. As a result,  $\text{dist}_0$  is not independent of the parametrization, even for the same curve. Thus, LAP does **not** compute a shape distance.

This problem can be resolved by only allowing uniform parameterizations of curves. Nonetheless, this might constrain the choice of possible matchings rather dramatically.

Another disadvantage of LAP is that it does not smoothly map one contour onto the other. In other words,  $m: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is just a bijection and not a homeomorphism or diffeomorphism.

Our goal is it now to define a matching energy that only considers diffeomorphic matchings  $m: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ . In addition, the minimum of such an energy should give rise to a semi-metric for shapes.

## Diffeomorphic Matching

Given two feature representations  $f_1, f_2: \mathbb{S}^1 \rightarrow \mathbb{R}^k$ , we want to define a matching energy that provides us with the same minimal value for  $g_1 := f_1 \circ \varphi$  and  $g_2 := f_2 \circ \varphi$  given a diffeomorphic reparameterization  $\varphi: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ .

If  $m$  is the optimal matching between  $f_1$  and  $f_2$ , we would expect that  $\tilde{m} := \varphi^{-1} \circ m \circ \varphi$  is the optimal matching between  $g_1$  and  $g_2$ .

For the previously defined  $E_0$  we have

$$\begin{aligned} \int_{\mathbb{S}^1} \text{dist}_{\mathcal{F}}(g_1(s), g_2 \circ \tilde{m}(s)) ds &= \int_{\mathbb{S}^1} \text{dist}_{\mathcal{F}}(f_1(\varphi(s)), f_2 \circ m(\varphi(s))) ds \\ &= \int_{\varphi(\mathbb{S}^1)} \text{dist}_{\mathcal{F}}(f_1(\varphi \circ \varphi^{-1}(s)), f_2 \circ m(\varphi \circ \varphi^{-1}(s))) \cdot \dot{\varphi}(\varphi^{-1}(s))^{-1} ds \\ &= \int_{\mathbb{S}^1} \text{dist}_{\mathcal{F}}(f_1(s), f_2 \circ m(s)) \cdot \dot{\varphi}(\varphi^{-1}(s))^{-1} ds \end{aligned}$$



### Geometrically Motivated Distance

Let us assume that two contours  $C_1, C_2 \subset \mathbb{R}^2$  together with their diffeomorphic parametrization  $c_i: \mathbb{S}^1 \rightarrow C_i$  are given. Further let  $f_i: \mathbb{S}^1 \rightarrow \mathbb{R}^k$  be their feature representation.

Then we define the torus  $T := C_1 \times C_2$  and the cost function

$$D: T \rightarrow \mathbb{R}_0^+ \quad (x, y) \mapsto \text{dist}_{\mathcal{F}}(f_1 \circ c_1^{-1}(x), f_2 \circ c_2^{-1}(y))$$

Note that neither  $T$  nor  $D$  depend on the specific parametrization  $c_1$  or  $c_2$ .

Given a matching  $m: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ , we define the 1D manifold

$$\Gamma(m) := \{(x, y) \in T \mid m \circ c_1^{-1}(x) = c_2^{-1}(y)\}$$

and a new energy

$$E_1^{(C_1, C_2)}(m) = \int_{\Gamma(m)} D(s) ds.$$

## Line Integral

Given a contour  $\Gamma \subset \mathbb{R}^N$  and a scalar function  $f: \Gamma \rightarrow \mathbb{R}$ , we would like to define the line integral  $\int_{\Gamma} f(s) ds$ . To this end, let us assume that we have a diffeomorphic coordinate map  $c: [0, 1] \rightarrow \Gamma$ .

Then, we can define the line integral as

$$\begin{aligned}\int_{\Gamma} f(s) ds &= \lim_{N \rightarrow \infty} \sum_{i=1}^N f \circ c \left( \frac{i}{N} \right) \left\| c \left( \frac{i}{N} \right) - c \left( \frac{i-1}{N} \right) \right\| \\ &= \int_0^1 f \circ c(t) \cdot \|c'(t)\| dt \\ &= \int_0^1 f \circ c(t) \cdot \sqrt{\det(c'(t)^\top c'(t))} dt\end{aligned}$$

## Diffeomorphic Matching

This definition leads to the following representation of the energy  $E_1$

$$\begin{aligned} E_1^{(C_1, C_2)}(m) &= \int_{\Gamma(m)} D(s) ds = \int_{\Gamma(m)} \text{dist}_{\mathcal{F}}(f_1 \circ c_1^{-1}(s_1), f_2 \circ c_2^{-1}(s_2)) ds \\ &= \int_{\mathbb{S}^1} \text{dist}_{\mathcal{F}}(f_1(t), f_2 \circ m(t)) \cdot \sqrt{\dot{c}_1(t)^2 + \frac{d}{dt}(c_2 \circ m)(t)^2} dt, \end{aligned}$$

which becomes for uniformly parameterized curves of same length  $2\pi$

$$E_1(m) = \int_{\mathbb{S}^1} \text{dist}_{\mathcal{F}}(f_1(t), f_2 \circ m(t)) \cdot \sqrt{1 + \dot{m}(t)^2} dt,$$

While this energy looks rather technical, it does not depend on the specific parameterizations of  $c_1$  and  $c_2$ .

## Properties of $\text{dist}_1$

Since only feature information is used, we can see that

$$\text{dist}_1(C_1, C_2) := \operatorname{argmin}_{m: \mathbb{S}^1 \rightarrow \mathbb{S}^1} E_1^{(C_1, C_2)}(m)$$

is a positive function defined on a shape space.

We obtain for a contour  $C$  that

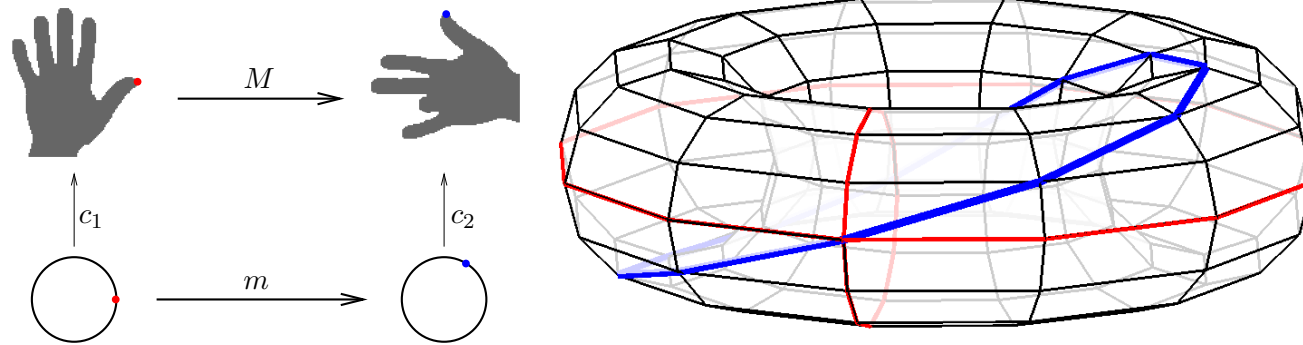
$$\text{dist}_1(C, C) \leq E_1^{(C, C)}(\text{id}) = 0.$$

Whether  $\text{dist}_1$  is positive definite depends on the chosen features.

For curvature,  $\text{dist}_1$  is positive definite.

Since, we always have  $E_1^{(C_1, C_2)}(m) = E_1^{(C_2, C_1)}(m^{-1})$ , we know that  $\text{dist}_1$  is symmetric. Thus,  $\text{dist}_1$  provides us with a semi-metric of our shape space.

## Matching Contour



A **shape matching** is a mapping  $M: \partial O_1 \rightarrow \partial O_2$  that maps corresponding boundary points onto one another. Thus, we assume that  $N$  points are selected from each contour.

We are interested in the **matching contour**  $\Gamma(m)$ , which can be described as a closed contour on the grid defined by  $N^2$  product nodes.

## Graph Representation

Let us assume that we have  $N$  ordered points  $x_0, \dots, x_{N-1} \in \mathbb{R}^2$  of the first contour  $C_1$  and  $N$  ordered points  $y_0, \dots, y_{N-1} \in \mathbb{R}^2$  of the second contour. In addition, we have the distance of the features stored in  $D \in \mathbb{R}^{N \times N}$ , i.e.,  $d_{ij} = \text{dist}_{\mathcal{F}}(f_1 \circ c_1^{-1}(x_i), f_2 \circ c_2^{-1}(y_j))$ .

Now we define the graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  that discretizes the torus  $T$ :

$$\mathcal{V} = \{0, \dots, N-1\} \times \{0, \dots, N-1\}$$

$$\mathcal{E} = \{[(i, j), (i \oplus 1, j)] \mid (i, j) \in \mathcal{V}\} \cup$$

$$\{[(i, j), (i, j \oplus 1)] \mid (i, j) \in \mathcal{V}\} \cup$$

$$\{[(i, j), (i \oplus 1, j \oplus 1)] \mid (i, j) \in \mathcal{V}\}$$

(horizontal edges)

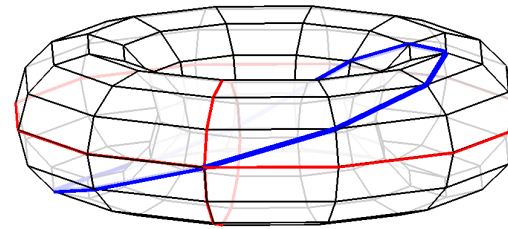
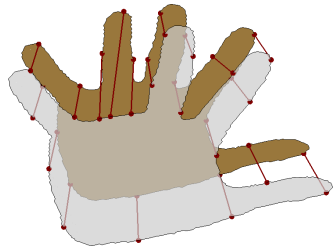
(vertical edges)

(diagonal edges),

where  $a \oplus b := (a + b) \bmod N$ .

In a last step we need to define a weight function  $w: \mathcal{E} \rightarrow \mathbb{R}$  that encodes our energy function  $E_1$ .

## Edge Weights



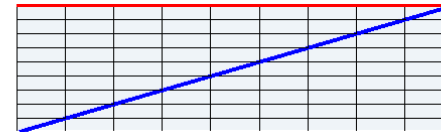
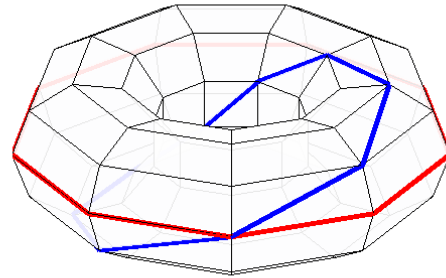
The optimal  $m: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  shall minimize the energy  $E_1(m) := \int_{\Gamma(m)} D(s) ds$ .

Since every edge in  $\mathcal{G}$  corresponds to a potential subset of  $\Gamma(m)$ , we define

$$w((i_1, j_1), (i_2, j_2)) := \frac{D_{i_1, j_1} + D_{i_2, j_2}}{2} \sqrt{\|v_1 - u_1\|^2 + \|v_2 - u_2\|^2} \\ \approx \int_{\vec{uv}} D(s) ds,$$

where  $u = (u_1, u_2) = (x_{i_1}, y_{j_1})$  and  $v = (v_1, v_2) = (x_{i_2}, y_{j_2})$ .

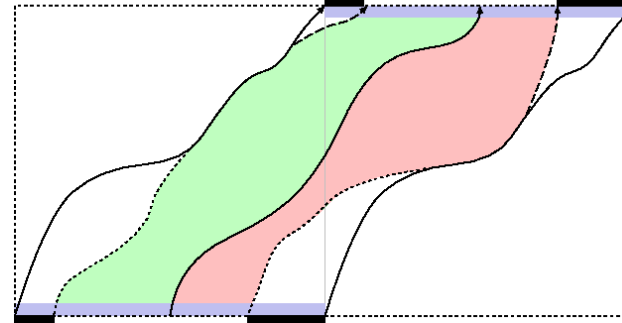
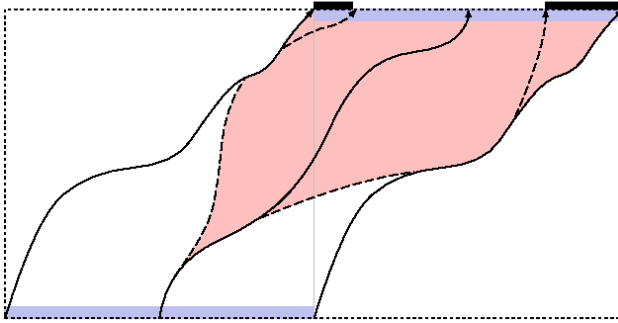
Dynamic Time Warping



1. Guess an initial correspondence  $c_1(x)$  on Shape 1 and  $c_2(y)$  on Shape 2.
  2. Cut the torus open along the curves  $\{x\} \times \mathbb{S}^1$  and  $\mathbb{S}^1 \times \{y\}$ .
  3. Find the shortest path between  $(x, y)$  and  $(x + 2\pi, y + 2\pi)$ .
- Step 3 can be done efficiently using dynamic time warping.  $\mathcal{O}(N^2)$
  - Iterating over initial correspondences slows the method down.  $\mathcal{O}(N^3)$

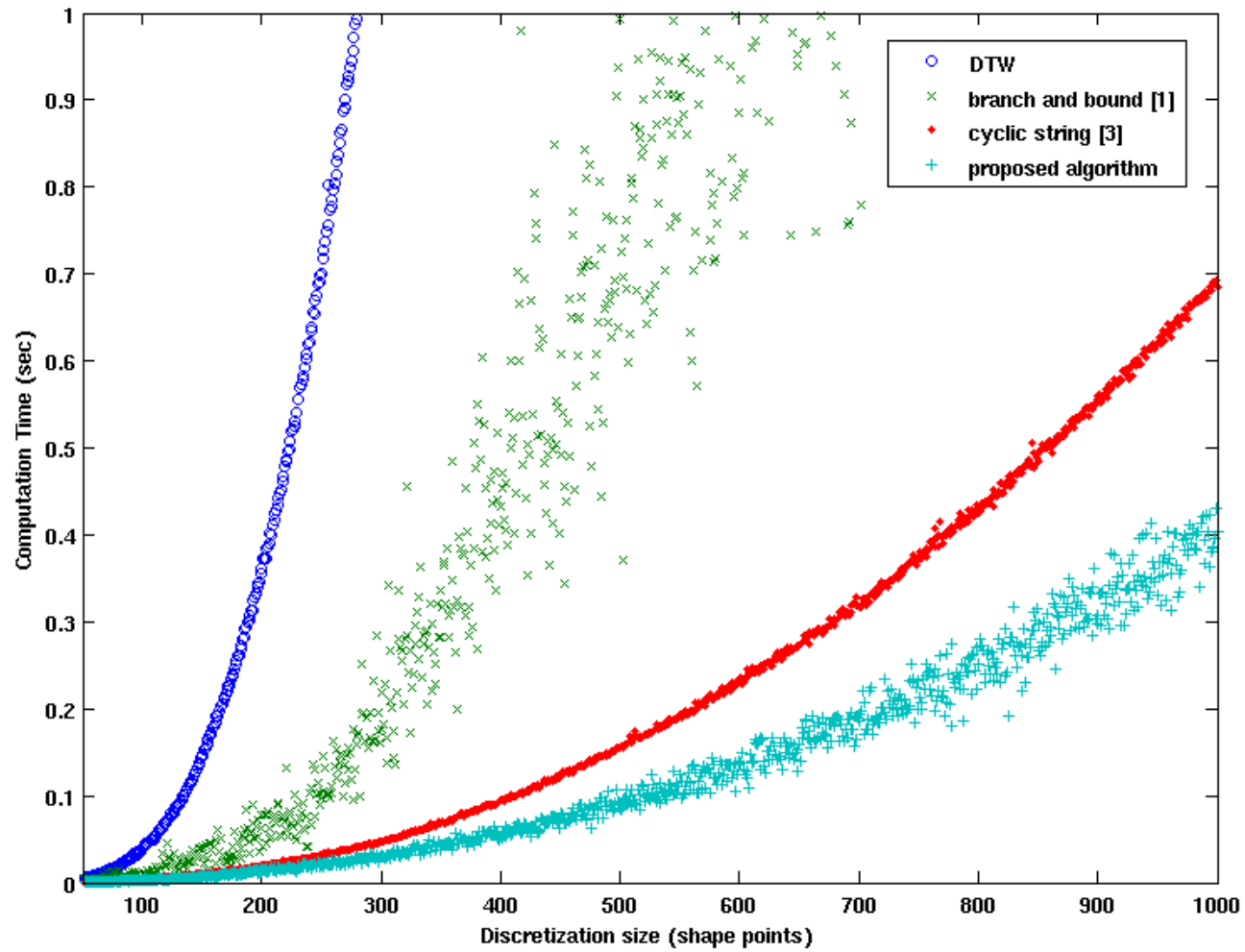


## Matching in Subcubic Runtime



1. Iteratively, divide each *searching region* into two regions.
2. Compute the shortest path for the *boundary regions* independently.
3. This leads to a better *region division*.

## Runtime





## Literature

### Applications of Dynamic Time Warping

- Sankoff and Kruskal, *Time Warps, String Edits and Macromolecules: The Theory and Practice of Sequence Comparison*, 1983, Addison-Wesley, Reading, MA.
- Geiger et al., *Dynamic programming for detecting, tracking and matching deformable contours*, 1995, IEEE PAMI 17 (3), 294–302.

### Matching as Shortest Circular Path

- Maes, *On a cyclic string-to-string correction problem*, 1990.
- Schmidt et al., *Fast Matching of Planar Shapes in Sub-cubic Runtime*, 2007, IEEE ICCV.