Convex Optimization for Computer Vision
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Weekly Exercises 8

Room 01.09.014
Friday, 17.6.2016, 09:00-11:00
Submission deadline: Wednesday, 15.6.2016, 14:15, Room: 02.09.023

Theory: Monotone Operators (0+12 Points)

Exercise 1 (4 Points)**.** Show that if T is nonexpansive and $\text{dom}(T) = \mathbb{R}^n$, then its set of fixed points

$$C := \left\{x \in \text{dom}(T) : x = Tx = (I - T)^{-1}(0)\right\},$$

is closed and convex. Additionally show that if T is a contraction, then T has a unique fixed point. (You may assume existence of a fixed point).

Solution. Since T is nonexpansive it is single valued. Let $x \neq y$ be fixed points i.e. $Tx = x$ and $Ty = y$, and define $z := \lambda x + (1 - \lambda)y$ for $\lambda \in (0, 1)$. Observe that,

$$\begin{aligned}
\|x - y\| &= \|x - Tz + Tz - y\| \\
&= \|Tx - Tz + Tz - Ty\| \\
&\leq \|x - Tz\| + \|Tz - y\| \\
&\leq \|x - z\| + \|z - y\| \\
&= (1 - \lambda)\|x - y\| + \lambda\|x - y\| = \|x - y\|,
\end{aligned}$$

with equality. From $\|Tx - Tz + Tz - Ty\| = \|x - y\|$ we conclude that $\exists\, \mu \in (0, 1)$ so that

$$Tz = \mu x + (1 - \mu)y.$$

Since

$$\begin{aligned}
(1 - \mu)\|x - y\| &= \|x - Tz\| \leq (1 - \lambda)\|x - y\| \\
\mu\|x - y\| &= \|Tz - y\| \leq \lambda\|x - y\|
\end{aligned}$$

we have $\mu = \lambda$, so C is convex. Since T is Lipschitz, T it is particularly continuous and therefore $I - T$ as a sum of two continuous functions is continuous as well. As C is the preimage of the closed set $\{0\}$ under the function $I - T$, it is closed as well. Let T be a contraction. Suppose there exist two fixed points $x \neq y$ with $Tx = x$ and $Ty = y$. Then

$$\|x - y\| = \|Tx - Ty\| < \|x - y\|,$$

a contradition. Therefore T has a unique fixed point.

Exercise 2 (4 Points)**.** Show that for closed, proper and convex $G : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, $F^* : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ the operator $T \subset \mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$ defined as

$$T := \begin{pmatrix} \partial G & K^T \\ -K & \partial F^* \end{pmatrix},$$

is monotone.

Solution. Let $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^{m+n}$. We have

$$\begin{aligned}
\langle Tu - Tv, u - v \rangle &= \left\langle \begin{pmatrix} \partial G u_1 + K^\top u_2 \\ -K u_1 + \partial F^* u_2 \end{pmatrix} - \begin{pmatrix} \partial G v_1 + K^\top v_2 \\ -K v_1 + \partial F^* v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle \\
&= \langle \partial G u_1 + K^\top u_2 - \partial G v_1 - K^\top v_2, u_1 - v_1 \rangle \\
&\quad + \langle -K u_1 + \partial F^* u_2 + K v_1 - \partial F^* v_2, u_2 - v_2 \rangle \\
&= \underbrace{\langle \partial G u_1 - \partial G v_1, u_1 - v_1 \rangle}_{\geq 0 \text{ since } \partial G \text{ monotone}} + \underbrace{\langle \partial F^* u_2 - \partial F^* v_2, u_2 - v_2 \rangle}_{\geq 0 \text{ since } \partial F^* \text{ monotone}} \\
&\quad + \underbrace{\langle K^\top (u_2 - v_2), u_1 - v_1 \rangle - \langle K(u_1 - v_1), u_2 - v_2 \rangle}_{=0} \geq 0,
\end{aligned}$$

And therefore T monotone.

Exercise 3 (4 Points)**.** Show that for closed, proper convex $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ the following equality holds

$$(\partial E)^{-1} = \partial E^*.$$

Solution.

$$\begin{aligned}
(x, y) \in (\partial E)^{-1} &\iff x \in \partial E(y) \\
&\iff y \in \partial E^*(x) \\
&\iff (x, y) \in \partial E^*
\end{aligned}$$

Programming: Cartooning (12 Points)

Exercise 4 (12 Points)**.** In this exercise your task is to compute a piecewise constant, cartoonish looking approximation of the input image. This can be done as follows: We begin selecting $k \ll 256$ different colors $\{c_1, c_2, \ldots c_k\}$ that are most present in the image, for example $c_1 =$ red, $c_2 =$ green, $c_3 =$ blue and $c_4 =$ yellow. We then segment the image into k disjoint regions, so that the overall boundary length is short and at the same time the pixels in the j-th region are close to the j-th color. Mathematically, one can solve the following optimization problem

$$\min_{u \in \mathbb{R}^{k \times n}} \iota_{\geq 0}(u) + \sum_{i=1}^{n} \sum_{j=1}^{k} u_{ij} f_{ij} + \alpha \sum_{j=1}^{k} \|Du^j\|_{2,1} \quad \text{s.t. } 1 \leq i \leq n \sum_{j=1}^{k} u_{ij} = 1,$$

where f_{ij} is given as the Euclidean distance of pixel i to color j and $u^j \in \mathbb{R}^n$ is the j-th row of u. Let $\tilde{u}$ be a minimizer of the problem above. Your final solution $\bar{u} \in \mathbb{R}^n$ is then given as

$$\bar{u}_i := c_m \quad \text{where } m := \arg\max_j \tilde{u}_{ij}.$$

Transform the problem above into a saddle point problem, i.e. identify F, G and the linear operator K, derive the proximal operators and solve it with PDHG.
Hints:

- Vectorize your problem, i.e. $u \in \mathbb{R}^{kn}$. Then the term $\sum_{i=1}^n \sum_{j=1}^k u_{ij} f_{ij}$ is just a scalar product $\langle u, f \rangle$.

- How to incorporate linear constraints:

$$\min_u G(u) + \tilde{F}(\tilde{K}u) \quad \text{s.t. } Au = b$$

  is equivalent to

$$\min_u G(u) + F(Ku) \text{ with } K := \begin{pmatrix} \tilde{K} \\ A \end{pmatrix} \text{ and } F(x) := \tilde{F}(x_1) + \iota_{=b}(x_2)$$

- Show that the linear operator of the segmentation problem can be chosen as:

$$K := \begin{pmatrix} D & & & \\ & D & & \\ & & \ddots & \\ & & & D \\ I & I & \dots & I \end{pmatrix} \in \mathbb{R}^{2nk+n \times kn},$$

  where D is a forward difference gradient operator for gray value images, that we had on the past sheets. It might be helpful to reason about the dimension of the operator considering the second bullet point.

- You may use MATLAB `kmeans` to find k representative colors of the input image.