Chapter 3

Duality

Convex Optimization for Computer Vision

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Summary: descent methods

For energies of the form

$$u^* \in \arg\min_{u \in \mathbb{R}^n} E(u),$$

for $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ proper, closed, convex, we discussed

Gradient descent:

- $\text{dom } E = \mathbb{R}^n$
- For $E \in \mathcal{F}_L^{1,1}(\mathbb{R}^n)$ energy convergence in $\mathcal{O}(1/k)$
- For $E \in \mathcal{S}_{m,L}^{1,1}(\mathbb{R}^n)$ energy and iterate convergence in $\mathcal{O}(c^k)$

Subgradient descent:

- $\text{dom}(E) = \mathbb{R}^n$
- Applicable to any Lipschitz-continuous convex energy
- Usually rather slow

Gradient projection: Generalizes gradient descent to arbitrary (nonempty, closed, convex) $\text{dom}(E)$.

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How powerful is the gradient projection algorithm?

Consider the total variation denoising problem

$$u^* \in \operatorname*{argmin}_u \frac{1}{2}\|u - f\|_2^2 + \alpha\|Du\|_{2,1},$$

with the finite difference operator $D : \mathbb{R}^{n\times m\times c} \to \mathbb{R}^{nm\times 2c}$.

Is subgradient descent really the best we can do despite the "nice" strongly convex energy?

Let's try something crazy to try to find a better algorithm:

$$\|g\| = \max_{|q|\leq 1}\langle q, g\rangle$$

Following the crazy idea...

The previous simple observation tells us that

$$
\begin{aligned}
\|g\|_{2,1} &= \sum_i \|g_i\| = \sum_i \max_{|q_i|\leq 1} \langle q_i, g_i\rangle \\
&= \max_{|q_i|\leq 1} \underbrace{\sum_i \langle q_i, g_i\rangle}_{=:\langle g,q\rangle} \\
&= \max_{\max_i \|q_i\|\leq 1} \langle g, q\rangle = \max_{\|q\|_{2,\infty}\leq 1} \langle g, q\rangle
\end{aligned}
$$

We may write

$$
\begin{aligned}
\min_u \frac{1}{2}\|u-f\|_2^2 + \alpha\|Du\|_1 &= \min_u \frac{1}{2}\|u-f\|_2^2 + \alpha \max_{\|q\|_{2,\infty}\leq 1} \langle Du, q\rangle \\
&= \min_u \max_{\|q\|_{2,\infty}\leq 1} \frac{1}{2}\|u-f\|_2^2 + \alpha\langle Du, q\rangle
\end{aligned}
$$

Can we switch min and max?

TV Minimization

Saddle point problems[1]

Let C and D be non-empty closed convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let S be a continuous finite concave-convex function on $C \times D$. If either C or D is bounded, one has

$$\inf_{v \in D} \sup_{q \in C} S(v,q) = \sup_{q \in C} \inf_{v \in D} S(v,q).$$

We can therefore compute

$$\begin{aligned} \min_u \frac{1}{2}\|u-f\|_2^2 + \alpha\|Du\|_1 &= \min_u \max_{\|q\|_{2,\infty} \leq 1} \frac{1}{2}\|u-f\|_2^2 + \alpha\langle Du, q\rangle \\ &= \max_{\|q\|_{2,\infty} \leq 1} \min_u \frac{1}{2}\|u-f\|_2^2 + \alpha\langle Du, q\rangle \end{aligned}$$

[1] Rockafellar, Convex Analysis, Corollary 37.3.2

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TV Minimization

Now the inner minimization problem obtains its optimum at

$$\begin{aligned} 0 &= u - f + \alpha D^* q, \\ \Rightarrow u &= f - \alpha D^* q. \end{aligned}$$

The remaining problem in q becomes

$$\begin{aligned} &\max_{\|q\|_{2,\infty}\leq 1} \frac{1}{2}\|f - \alpha D^* q - f\|_2^2 + \alpha\langle D(f - \alpha D^* q), q\rangle \\ =& \max_{\|q\|_{2,\infty}\leq 1} \frac{1}{2}\|\alpha D^* q\|_2^2 + \alpha\langle Df, q\rangle - \|\alpha D^* q\|_2^2 \\ =& \max_{\|q\|_{2,\infty}\leq 1} -\frac{1}{2}\|\alpha D^* q - f\|_2^2 \end{aligned}$$

TV Minimization

Since we prefer minimizations over maximizations, we write

$$\hat{q} = \underset{\|q\|_{2,\infty} \leq 1}{\operatorname{argmax}} -\frac{1}{2}\|\alpha D^* q - f\|_2^2$$

$$= \underset{\|q\|_{2,\infty} \leq 1}{\operatorname{argmin}} \frac{1}{2}\left\|D^* q - \frac{f}{\alpha}\right\|_2^2$$

This is a problem we know how to solve! An L-smooth function over a simple convex set: Gradient projection

$$q^{k+1} = \pi_C\left(q^k - \tau D\left(D^* q^k - \frac{f}{\alpha}\right)\right),$$

where $C = \{q \in \mathbb{R}^{nm \times 2c} \mid \|q\|_{2,\infty} \leq 1\}$.

A conceptual way to reformulate energy minimization problems?

Maybe our idea

$$\|g\| = \max_{|q|\leq 1} \langle q, g \rangle$$

was not so crazy but rather conceptual?

Definition: Convex Conjugate

We define the *convex conjugate* of the function $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ to be

$$E^*(p) = \sup_{u \in \mathbb{R}^n} \left(\langle u, p \rangle - E(u) \right).$$

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Convex conjugates

Convexity of the Convex Conjugate

The convex conjugate E^* of any proper function $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex.

Proof: Board

Convexity of the Convex Conjugate

The convex conjugate E^* of any proper function $E : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is closed.

Proof: Linear functions are closed and arbitrary intersections of closed sets are closed.

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