Nonnegativity of 1-homogeneous functions

$J(0) = 0$ $J(0) = 0$ is obvious. Furthermore, note that

$$0 = J\left(\frac{1}{2}u + \frac{1}{2}(-u)\right) \leq \frac{1}{2}(J(u) + J(-u)) = J(u)$$

hold for all u.

Subdifferential of 1-homogeneous functions

The inclusion

$$\{p \in \mathbb{R}^n \mid J(u) = \langle p, u \rangle,\ J(v) \geq \langle p, v \rangle\ \forall v \in \mathbb{R}^n\} \subset \partial J(u)$$

is obvious. Let $p \in \partial J(u)$. Then

$$J(v) - J(u) - \langle p, v - u \rangle \geq 0$$

holds for all v. We choose $v = 0.5u$ and $v = 2u$, and use the 1-homogeneity of J to conclude that $J(u) = \rangle p, u \rangle$. The remaining inequality follows from the definition of the subdifferential.

0-homogeneous subdifferential

If

$$J(v) - J(u) - \langle p, v - u \rangle \geq 0$$

holds for all v then

$$J(v) - \frac{1}{a}J(au) - \frac{1}{a}\langle p, av - au \rangle \geq 0$$

holds for any $a > 0$. We multiply the inequality by a to obtain

$$J(av) - J(au) - \langle p, av - au \rangle \geq 0$$

for all v, which is equivalent to

$$J(v) - J(au) - \langle p, v - au \rangle \geq 0$$

for all v, i.e. $p \in \partial J(au)$.

Kernel of 1-homogeneous functions

Let $u, v \in \ker(J)$ and $a, b \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} J(au + bv) &= (|a| + |b|)J\left(\frac{|a|}{|a| + |b|}\operatorname{sign}(a)u + \frac{|b|}{|a| + |b|}\operatorname{sign}(b)v\right) \\ &\leq |a|J(\operatorname{sign}(a)u) + |b|J(\operatorname{sign}(b)v) = 0 \end{aligned}$$

Domain of 1-homogeneous functions

Same trick as above with $< \infty$ instead of $= 0$.

Ground states exist

Without restriction of generality $\mathrm{dom}(J) = \mathbb{R}^n$, such that J is continuous. Remember that a convex function on $\mathbb{R}^n$ is continuous in the interior of the domain. A ground state is defined by

$$\min_{u \in M} J(u)$$

with

$$M = \{u \in \mathbb{R}^n \mid \|u\|_2 = 1, u \in \ker(J)^\perp\}.$$

Since M is non-empty, bounded and closed, i.e. compact, the minimum is attained.

Ground states are singular vectors

Note that

$$\langle \lambda_0 u_0, u_0 \rangle = \lambda_0 \|u_0\|^2 = \lambda_0 = J(u_0)$$

holds. Additionally, for any $0 \neq v \in \mathbb{R}^n$, we find

$$\begin{aligned} \langle \lambda_0 u_0, v \rangle &= \lambda_0 \|v\|_2 \left\langle u_0, \frac{v}{\|v\|_2} \right\rangle \\ &\leq \lambda_0 \|v\|_2 = J(u_0)\|v\|_2. \end{aligned}$$

Now by the definition of ground states, we have

$$J(u_0) \leq J\left(\frac{v}{\|v\|_2}\right) = \frac{1}{\|v\|_2} J(v),$$

such that we can conclude

$$\langle \lambda_0 u_0, v \rangle \leq J(u_0)\|v\|_2 \leq J(v).$$

Using the characterization of the subdifferential of 1-homogeneous functionals we find that

$$\lambda_0 u_0 \in \partial J(u_0),$$

which shows that u_0 is a singular vector.