Analysis of Three-Dimensional Shapes
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# Weekly Exercises 6

Room: 02.09.023

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Submission deadline: Tue, 09.06.2015, 23:59 to windheus@in.tum.de

Please send in only Latex-PDF. If you have hand-written solutions, please hand them in during the lecture.

## Laplacian

**Exercise 1** (One Point)**.** In this exercise we investigate the eigenvectors of the Laplace matrix $L = M^{-1}C \in \mathbb{R}^{n\times n}$ as introduced in the lecture and last exercise. (In the last exercise the stiffness (or cotangent) matrix $C$ was denoted by $S$.)

1. Show that $\phi$ is an eigenvector of $L$ with eigenvalue $\lambda$ iff it is a solution to the generalized eigenvalue problem

$$\lambda M\phi = C\phi$$

2. Show that $\langle\cdot,\cdot\rangle_M := \langle\cdot, M\cdot\rangle$ defines an inner product.

3. Show that the Laplacian matrix $L$ is symmetric with respect to $\langle\cdot,\cdot\rangle_M$, i.e. $\langle Lx, y\rangle_M = \langle x, Ly\rangle_M$.

4. Show that $L$ has real eigenvalues?

5. Show that you can find eigenvectors $\{\phi_i\}$ of $L$ such that $\Phi^T M\Phi = \mathrm{Id}$. Here $\Phi$ is the matrix with the eigenvectors as columns

$$\Phi = \begin{pmatrix} | & & | \\ \phi_1 & \dots & \phi_n \\ | & & | \end{pmatrix}.$$

6. Let $f \in \mathbb{R}^n$, define coefficients $\alpha_i \in \mathbb{R}$ by $\alpha_i = \langle f, \phi_i\rangle_M$. Show that $f = \sum_i \alpha_i\phi_i$, i.e. $\{\phi_i\}$ is an orthonormal basis of $\mathbb{R}^n$.

**Solution.** 1. Let $\phi$ is be an eigenvector of $L$ with eigenvalue $\lambda$, it holds by the definition

$$\begin{aligned} \lambda\phi = L\phi &= M^{-1}C\phi \\ \Leftrightarrow \lambda M\phi &= C\phi. \end{aligned}$$

2. We need to show that $M$ is positive definite. $M$ is defined by

$$M_{i,j} = \begin{cases} \frac{A(T_1)+A(T_2)}{12} & \text{if } i \neq j \text{ and } T_1, T_2 \text{ are common triangles,} \\ \sum_{T \in N(i)} \frac{A(T)}{6} & \text{if } i = j \text{ and } N(i) \text{ is the set of incident triangles,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $n$ be the number of vertices and $m$ be the number of edges. We now construct matrix $F \in \mathbb{R}^{m \times n}$, such that $M = F^\top F$. Let the $e$-th edge connect vertices $i, j$, set $F_{e,i} = F_{e,j} = \sqrt{\frac{A(T_1)+A(T_2)}{12}}$, where $T_1, T_2$ are the common triangles of vertices $i, j$. Set all other entries of $F$ to zero. Calculating the matrix-matrix product $F^\top F$ we get $(F^\top F)_{i,j} = \sum_e F_{e,i} F_{e,j}$. Thus $(F^\top F)_{i,j} = \frac{A(T_1)+A(T_2)}{12}$ if $i, j$ are adjacent and $(F^\top F)_{i,j} = 0$ otherwise. If $i = j$ we get $(F^\top F)_{i,j} = \sum_e F_{e,i}^2 = \sum_{T \in N(i)} \frac{A(T)}{6}$. Thus $M = F^\top F$ and $m$ positive semi-definite. Since $M$ is invertible, $M$ is also positive definite.

3. Recall that $\langle Ax, y \rangle = \langle x, A^* y \rangle$ and for some scalar $\lambda \in \mathbb{C}$ it holds $\lambda \langle x, y \rangle = \langle x, \overline{\lambda} y \rangle$. Since $M, C$ are real and symmetric, we have $\langle Lx, y \rangle_M = \langle MLx, y \rangle = \langle Cx, y \rangle = \langle x, Cy \rangle = \langle x, MLy \rangle = \langle x, Ly \rangle_M$. Now let $\lambda \phi = L\phi$, we have $\lambda \langle \phi, \phi \rangle_M = \langle \lambda \phi, \phi \rangle_M = \langle L\phi, \phi \rangle_M = \langle \phi, L\phi \rangle_M = \langle \phi, \lambda \phi \rangle_M = \overline{\lambda} \langle \phi, \phi \rangle_M$. Since $\langle \phi, \phi \rangle_M > 0$ it holds $\lambda = \overline{\lambda}$.

4. Let $L\phi_1 = \lambda_1 \phi_1, L\phi_2 = \lambda_2 \phi_2, \|\phi_1\|_M = \|\phi_2\|_M = 1$. We get $\lambda_1 \langle \phi_1, \phi_2 \rangle_M = \langle \lambda_1 \phi_1, \phi_2 \rangle_M = \langle L\phi_1, \phi_2 \rangle_M = \langle \phi_1, L\phi_2 \rangle_M = \langle \phi_1, \lambda_2 \phi_2 \rangle_M = \lambda_2 \langle \phi_1, \phi_2 \rangle_M$. Thus $\langle \phi_1, \phi_2 \rangle_M = 0$, if $\lambda_1 \neq \lambda_2$ and $\langle \phi_1, \phi_2 \rangle_M = 1$, if $\lambda_1 = \lambda_2$ (ignoring eigenvalues of multiplicity $> 1$).

5. $\alpha_i = \langle f, \phi_i \rangle_M = \langle \phi_i, Mf \rangle$, thus $\alpha = \Phi^\top M f = \Phi^{-1} f$, thus $f = \Phi \alpha$.

# The Heat Equation

Let $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be a continuous sequence of vectors $u(t) \in \mathbb{R}^n$, where we call $t$ the time parameter. Since the eigenvectors $\{\phi_i\}$ of $L$ form an orthonormal basis of $\mathbb{R}^n$, $u(t)$ can be written as $u(t) = \sum_i \alpha_i(t) \phi_i$, where the coefficients $\alpha_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are real-valued functions of time parameter $t$. We say $u(t)$ is a discrete distribution of heat if it satisfies the *heat equation*

$$\frac{\partial}{\partial t} u(t) = Lu(t).$$

From the linearity of differentiation we can write $\frac{\partial}{\partial t} u(t)$ as

$$\frac{\partial}{\partial t} u(t) = \sum_i \phi_i \frac{\partial}{\partial t} \alpha_i(t).$$

**Exercise 2** (One Point)**.** Let $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n, u(t) = \sum_i \alpha_i(t) \phi_i$, now be any such sequence of vectors that satisfies the heat equation.

1. Show that there exists coefficients $c_i \in \mathbb{R}, 1 \leq i \leq n$, such that

$$\alpha_i(t) = c_i \exp(\lambda_i t).$$

2. Let $u_0 \in \mathbb{R}^n$ and set $u(0) = u_0$. Show that the coefficients $c_i$ can be computed by $c_i = \langle u_0, \phi_i \rangle_M$.

The exercise shows that if we let some initial heat distribution $u_0$ diffuse over time $t$ the resulting heat distribution $u(t)$ can be computed by

$$u(t) = \sum_i \langle u_0, \phi_i \rangle_M \exp(\lambda_i t) \phi_i.$$

If our triangle mesh has $n$ vertices we can define the *heat kernel signature* $\mathrm{HKS}(v_i, t)$ for $v_i$ at time $t$ by

$$\mathrm{HKS}(v_i, t) = \sum_i \langle e_i, \phi_i \rangle_M \exp(\lambda_i t) \phi_i,$$

where $e_i \in \mathbb{R}^n$ is the vector that is 0 everywhere except in the $i$-th component. Now we can discretize the time line by taking a finite subset $\mathcal{T} = \{t_1, \ldots, t_T\} \subset \mathbb{R}_{\geq 0}$ and define the vector valued heat kernel signature of vertex $v_i$ by

$$\mathrm{HKS}_{\mathcal{T}}(v_i) = \big(\mathrm{HKS}(v_i, t_1), \ldots, \mathrm{HKS}(v_i, t_T)\big).$$

**Solution.** 1. Let $u(t) = \sum_i \alpha_i(t)\phi_i$, we expand the heat equation on both sides:

$$\begin{aligned}
\frac{\partial}{\partial t} u(t) &= Lu(t) \\
\sum_i \phi_i \frac{\partial}{\partial t} \alpha_i(t) &= L(\sum_i \alpha_i(t)\phi_i) = \sum_i \alpha_i(t) L\phi_i \\
\sum_i \phi_i \frac{\partial}{\partial t} \alpha_i(t) &= \sum_i \phi_i \lambda_i \alpha_i(t).
\end{aligned}$$

Since vectors $\phi_i$ are linear independent, it holds for all $i$:

$$\frac{\partial}{\partial t} \alpha_i(t) = \lambda_i \alpha_i(t).$$

From basic analysis we know that $\alpha_i(t) = c_i \exp(\lambda_i t)$, where $c_i \in \mathbb{R}$ is any number, is a solution to this differential equation.

2. Since $\{\phi_i\}$ is an orthonormal basis of $\mathbb{R}^n$ it holds $u_0 = \sum_i \langle \phi_i, u_0 \rangle_M \phi_i$. Thus $\langle \phi_i, u_0 \rangle_M = \alpha_i(0) = c_i \exp(\lambda_i 0) = c_i$.

# Programming: The Discrete Laplace Operator

**Exercise 3** (Two points)**.** Download and expand the file `exercise6.zip` from the lecture website. Modifiy the files `cotanmatrix.m`, `massmatrix.m`, `heatsimulation.m`, and `hks.m` to implement the functions as explained below. You can run the script `exercise.m` to test and visualize your solution.

`cotanmatrix.m` The function should compute the matrix $C \in \mathbb{R}^{n \times n}$ based on the cotangent scheme as defined in the lecture. The triangle mesh is given by matrices $V \in \mathbb{R}^{n \times 3}$ and $F \in \mathbb{N}^{m \times 3}$, where $n$ is the number of vertices and $m$ the number of triangles. $C$ should be returned in sparse format.

`massmatrix.m` The function should compute the matrix $M \in \mathbb{R}^{n \times n}$ based on the scheme defined in the lecture. The triangle mesh is given by matrices $V \in \mathbb{R}^{n \times 3}$ and $F \in \mathbb{N}^{m \times 3}$, where $n$ is the number of vertices and $m$ the number of triangles. $M$ should be returned in sparse format.

`exercise.m` Look at the code for the eigen decomposition. You see it is very easy to compute the generalized eigen decomposition $\lambda M \phi = C \phi$ by the matlab function `eigs`.

`heatsimulation.m` Given some initial heat distribution $u_0 \in \mathbb{R}^n$, the function simulates the diffusion of heat on the mesh. The function should display the distribution of heat at several given time points $t_1, \ldots, t_T \in \mathbb{R}_+$.

`hks.m` The function should compute for each point on the mesh the heat kernel signature at time points $t_1, \ldots, t_T \in \mathbb{R}_+$.