Analysis of Three-Dimensional Shapes
(IN2238, TU München, Summer 2015)

Partial similarity
(22.06.2015)

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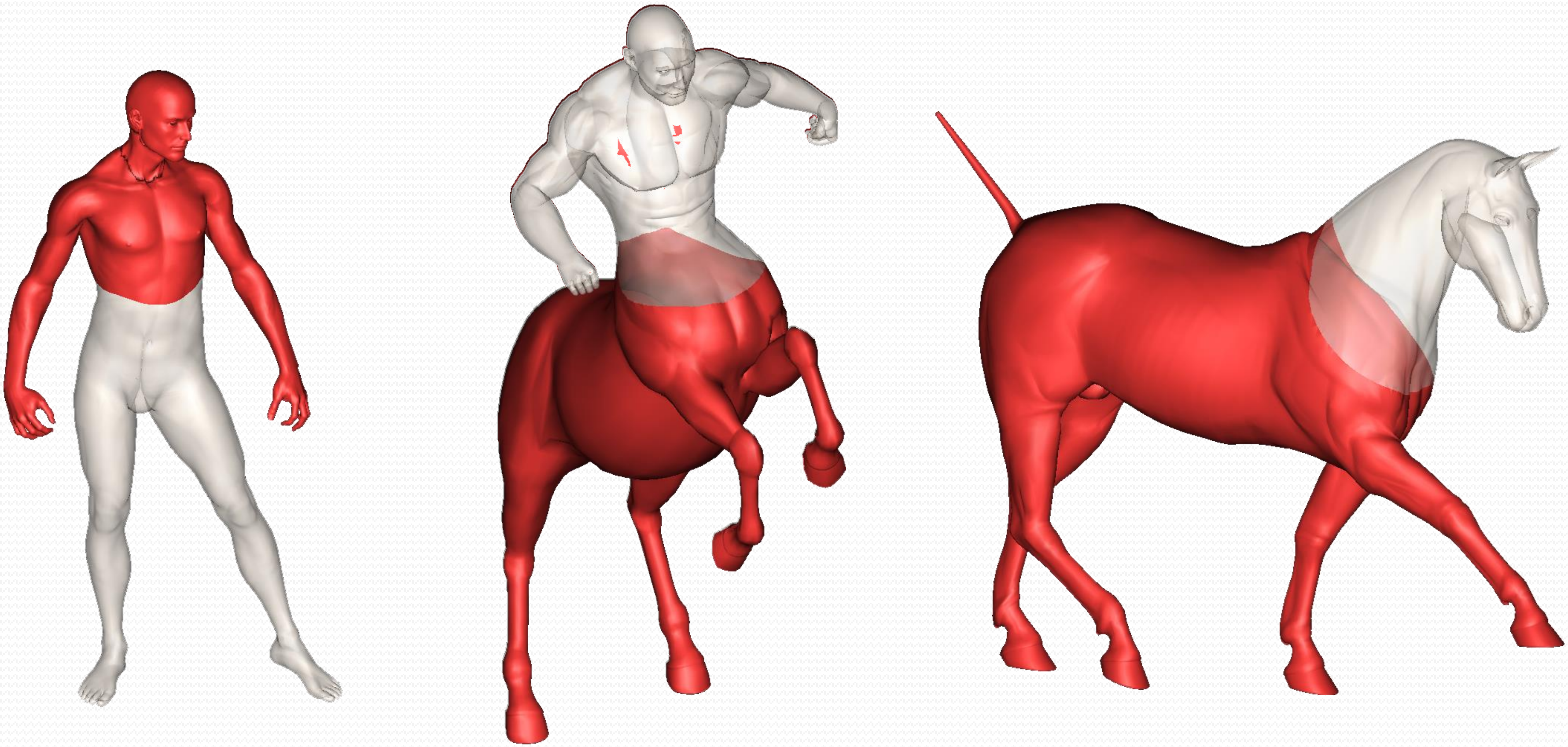
Partial similarity

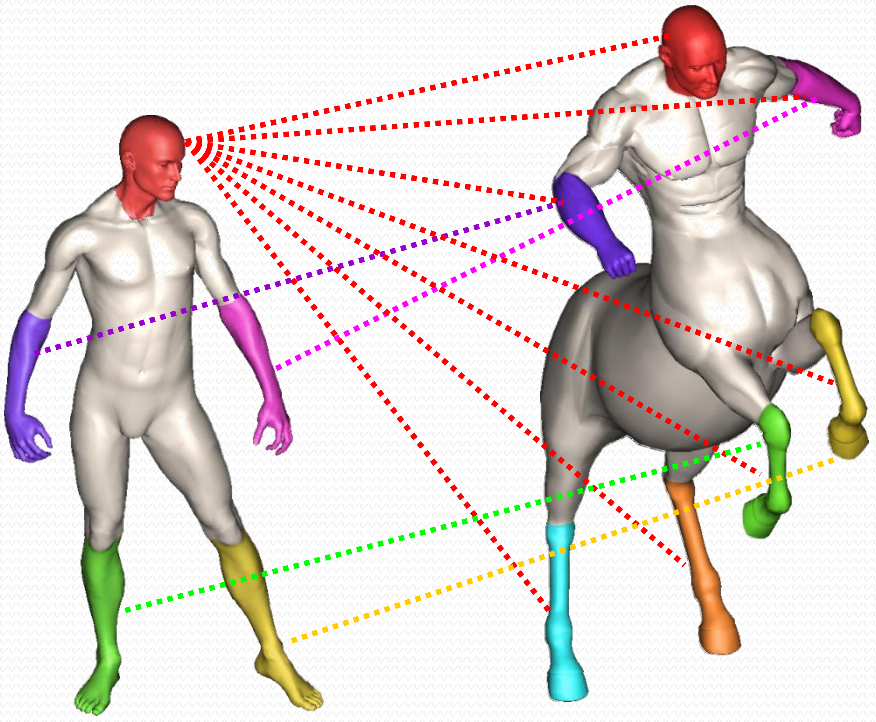


Partially human

Partially equine

Example

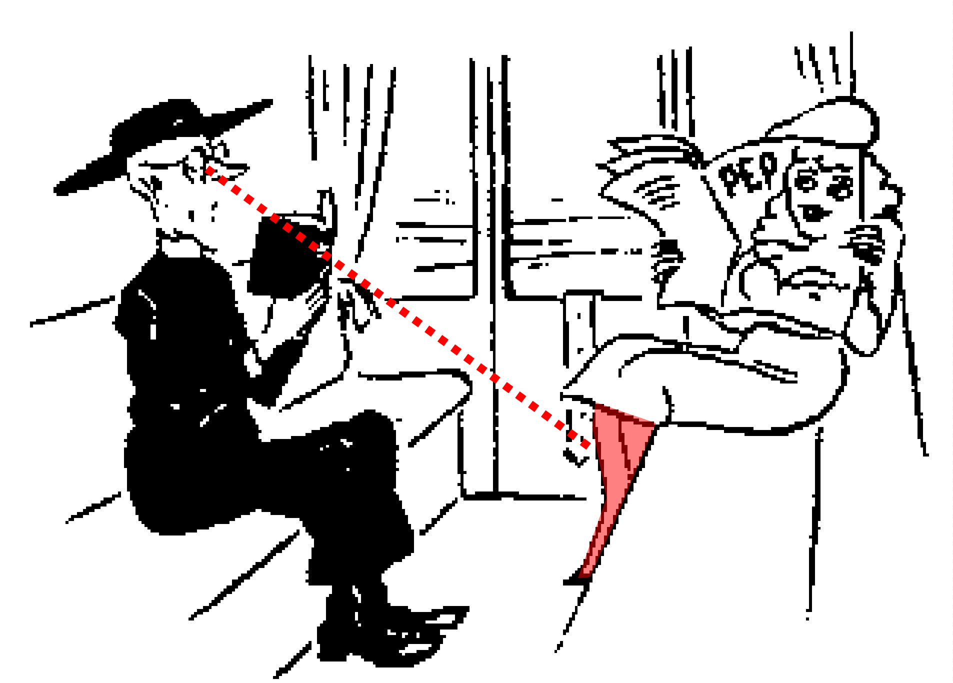
Recognition by parts



We can phrase the partial matching problem as a **segment-and-compare** problem. Each region is compared using classical **full similarity** criterion (e.g. Gromov-Hausdorff distance).

Significance

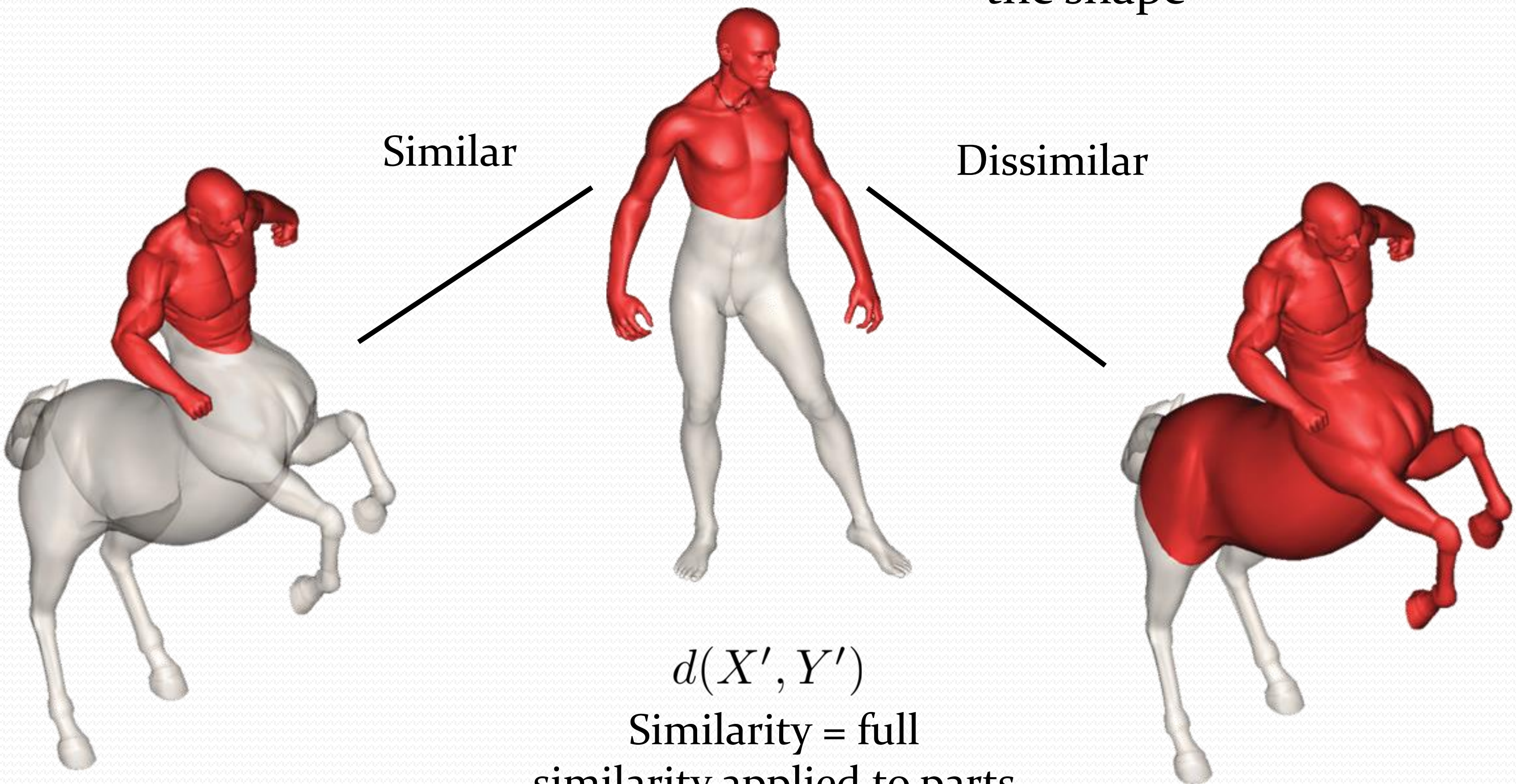
We will say that shapes *X* and *Y* are **partially similar** if they have **significant** similar parts.



Parts and similarity

$(X' \subseteq X \, , d_X|_{X'})$

A **part** is a subset of the shape



$d(X', Y')$

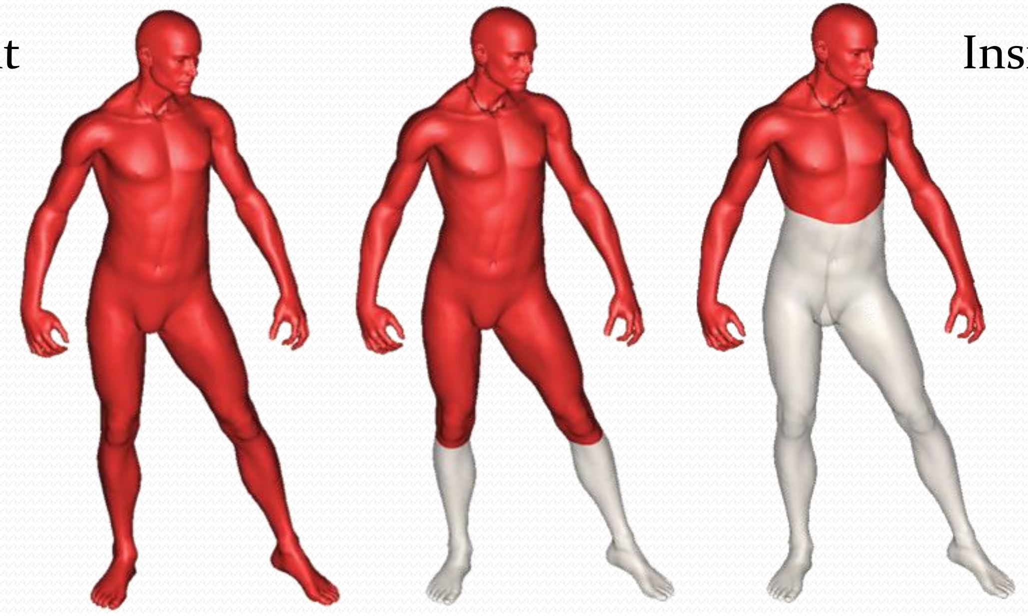
Similarity = full similarity applied to parts

Significance

Significance = measure
of the part (e.g. area)

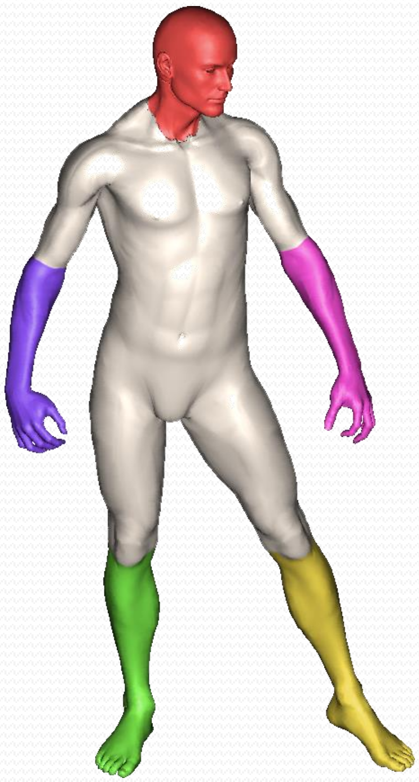
$$\mu(X')$$

Significant



Insignificant

Significance



Which one of the colored regions is the most significant part?

Significance is really a *semantic* definition, and in general it depends on the data.

Multi-criterion optimization

Let us consider the following optimization problem:

$$\min_{X' \subseteq X, Y' \subseteq Y} \left(d(X', Y') \,, \;\; -\mu(X') - \mu(Y') \right)$$

We want to simultaneously **minimize dissimilarity and insignificance** over all the possible pairs of parts. Note that the objective function is vector-valued (also called *vector optimization* or *multi-objective optimization*).

$$\min_{X' \subseteq X, Y' \subseteq Y} \left(d(X', Y') \,, \; s(X', Y') \right)$$

We will use $s(X', Y')$ to denote the insignificance $-(\mu(X') + \mu(Y'))$

Pareto optima

$\min_x f(x)$

A solution x^* is said to be a **global optimum** to the problem above, if there is no other *x* such that:

$$f(x) < f(x^*)$$

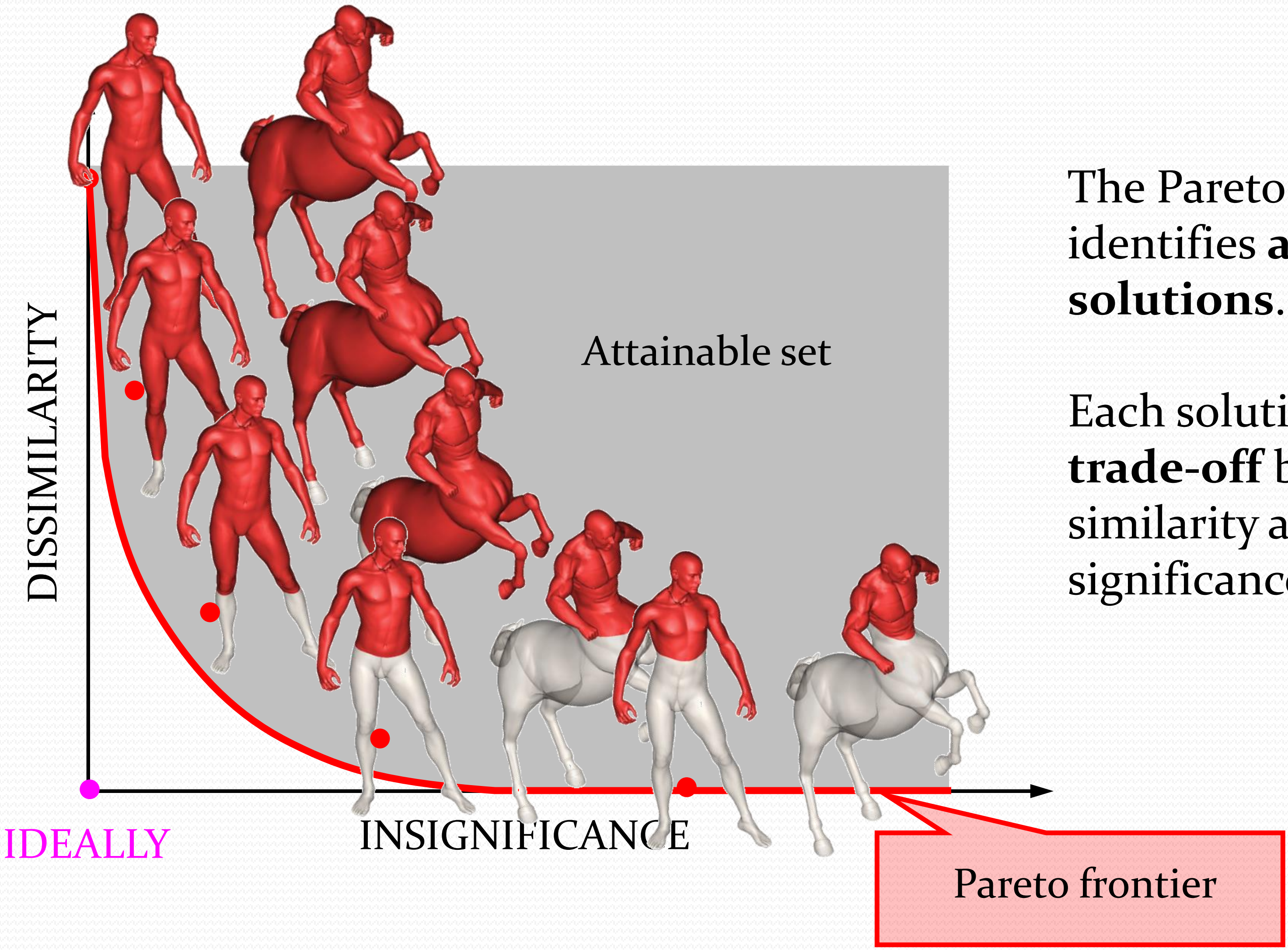
$\min_x (f_1(x), f_2(x), \ldots, f_K(x))$

A solution x^* is said to be a **Pareto optimum** to the problem above, if there is no other *x* such that:

$$\begin{gathered} f_1(x) < f_1(x^*) \\ \vdots \\ f_K(x) < f_K(x^*) \end{gathered}$$

An optimum is a solution such that there is no other better solution. In the multi-criterion case, “better” means that **all criteria are better**.

Pareto frontier



The Pareto frontier identifies **all optimal solutions**.

Each solution offers a **trade-off** between similarity and significance.

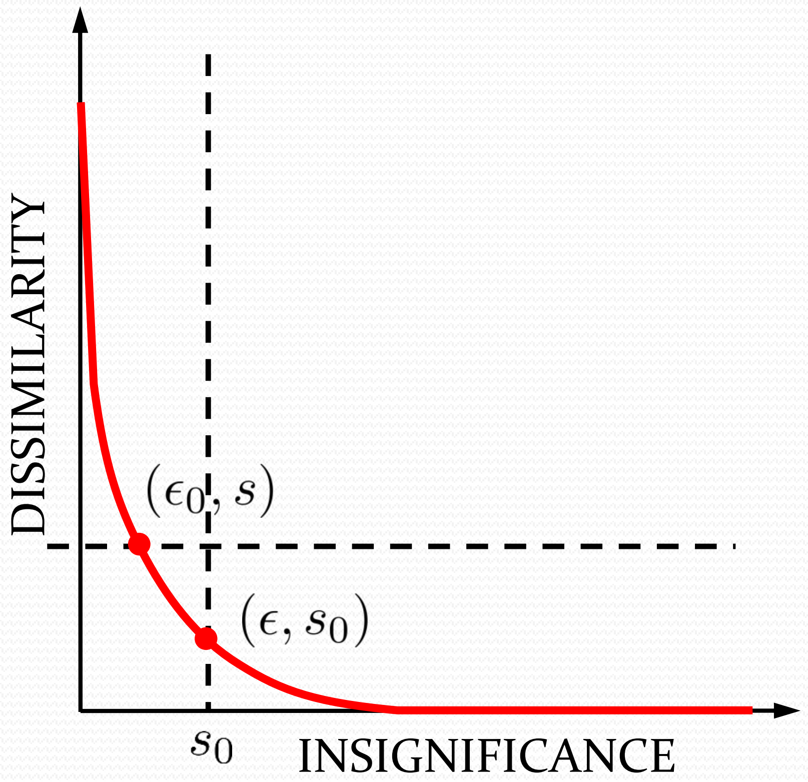
Scalarization

In practice, we can reformulate the partial similarity problem as a classical optimization problem by **fixing** a value of insignificance.

$$\min_{X' \subseteq X, Y' \subseteq Y} d(X', Y') \quad \text{s.t. } s(X', Y') \leq s_0$$

...or we can fix instead a value of dissimilarity:

$$\min_{X' \subseteq X, Y' \subseteq Y} s(X', Y') \quad \text{s.t. } d(X', Y') \leq \epsilon_0$$



Part indicator functions

Let us consider one of the two problems from the previous slide, for example:

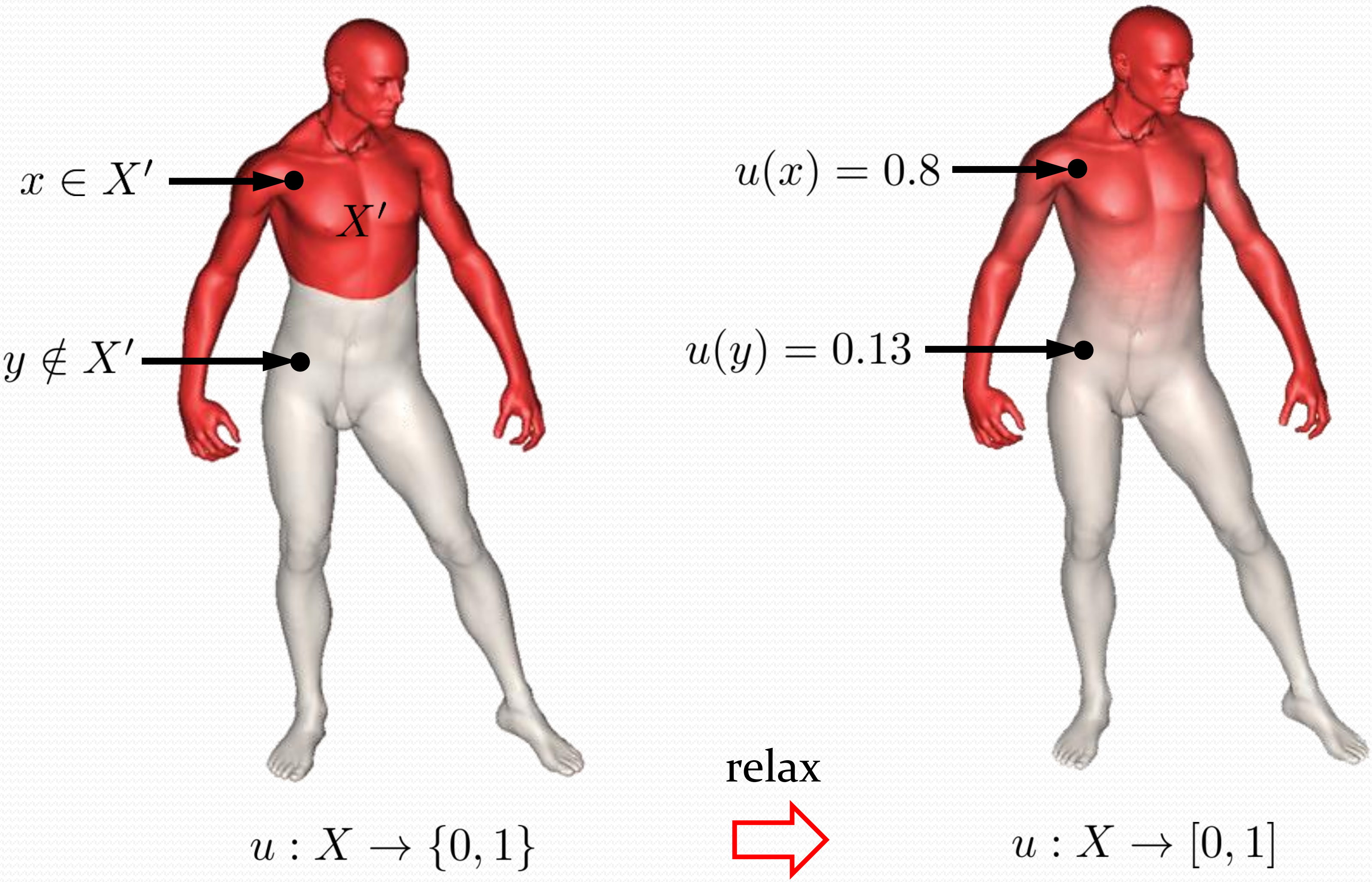
$$\min_{X' \subseteq X, Y' \subseteq Y} d(X', Y')$$
$$\text{s.t. } s(X', Y') \leq s_0$$

The problem requires us to optimize over all possible subsets of the two shapes. We will model these subsets as indicator functions:

$$u_{X'}(x) = \begin{cases} 1 & x \in X' \\ 0 & x \notin X' \end{cases}$$

Still, we see that we are dealing with a **combinatorial** optimization problem (which is in general *very hard* to solve).

Membership functions



Fuzzy partial similarity

Let us first pass to an unconstrained optimization problem:

$$\min_{X' \subseteq X, Y' \subseteq Y} d(X', Y') \quad \text{s.t. } s(X', Y') \leq s_0 \qquad \Rightarrow \qquad \min_{X' \subseteq X, Y' \subseteq Y} d(X', Y') - \alpha s(X', Y')$$

We now apply the “fuzzy” relaxation, and get:

$$\min_{\substack{u: X \to [0,1] \\ v: Y \to [0,1] \\ \phi: X \to Y}} d(u, v, \phi) - \alpha s(u, v)$$

$$\int \int |d_X(x, x') - d_Y(\phi(x), \phi(x'))| u(x) u(x') da(x) da(x')$$

dissimilarity (just the usual metric distortion, weighted by *u*)

$$-\alpha \left(\int_X u(x) da(x) + \int_Y v(y) da(y) \right)$$

insignificance

Optimization

1

Freeze parts and solve for correspondence

$$\min_{u,v,\varphi} D(u,v,\varphi) + \alpha s(u,v)$$

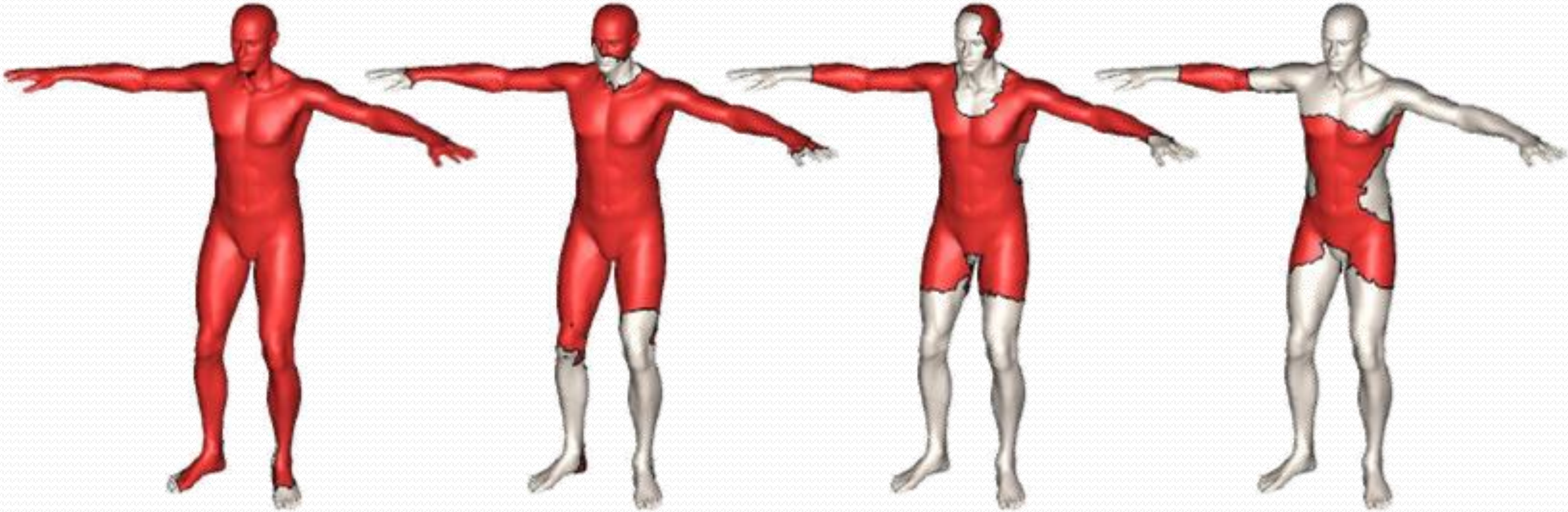
$$+ \text{const}$$

2

Freeze correspondence and solve for parts

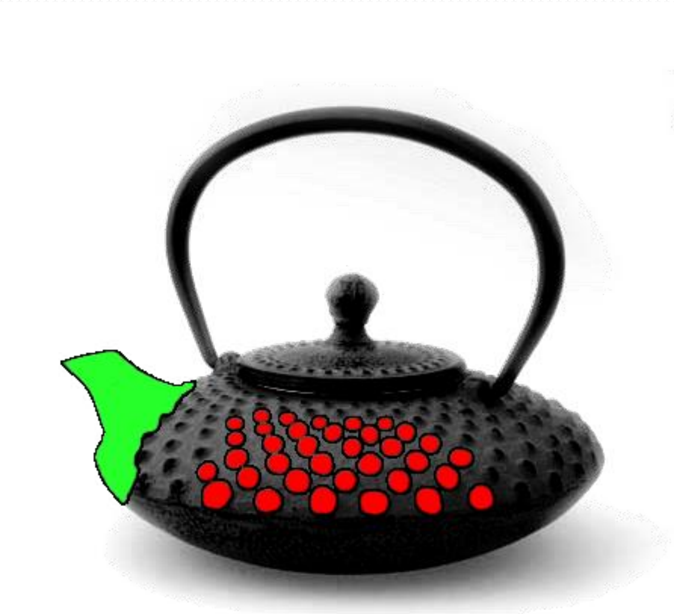
$$\min_{u,v,\varphi} D(u,v,\varphi) - \alpha s(u,v)$$

Not only size matters



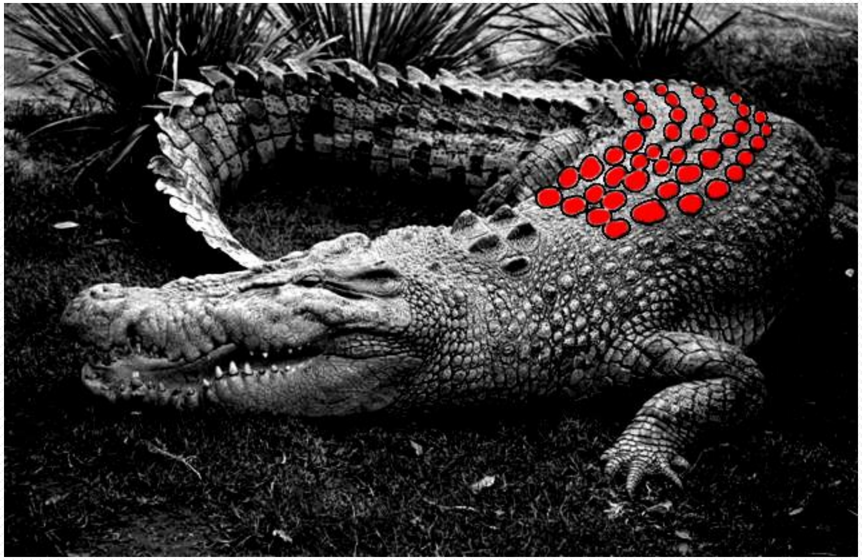
Part size is controllable by parameter α . However, there is nothing that constrains the resulting regions to be **regular**, or in other words to "stay together".

What is better?...



...or one large part?

Many small parts...



Regularity

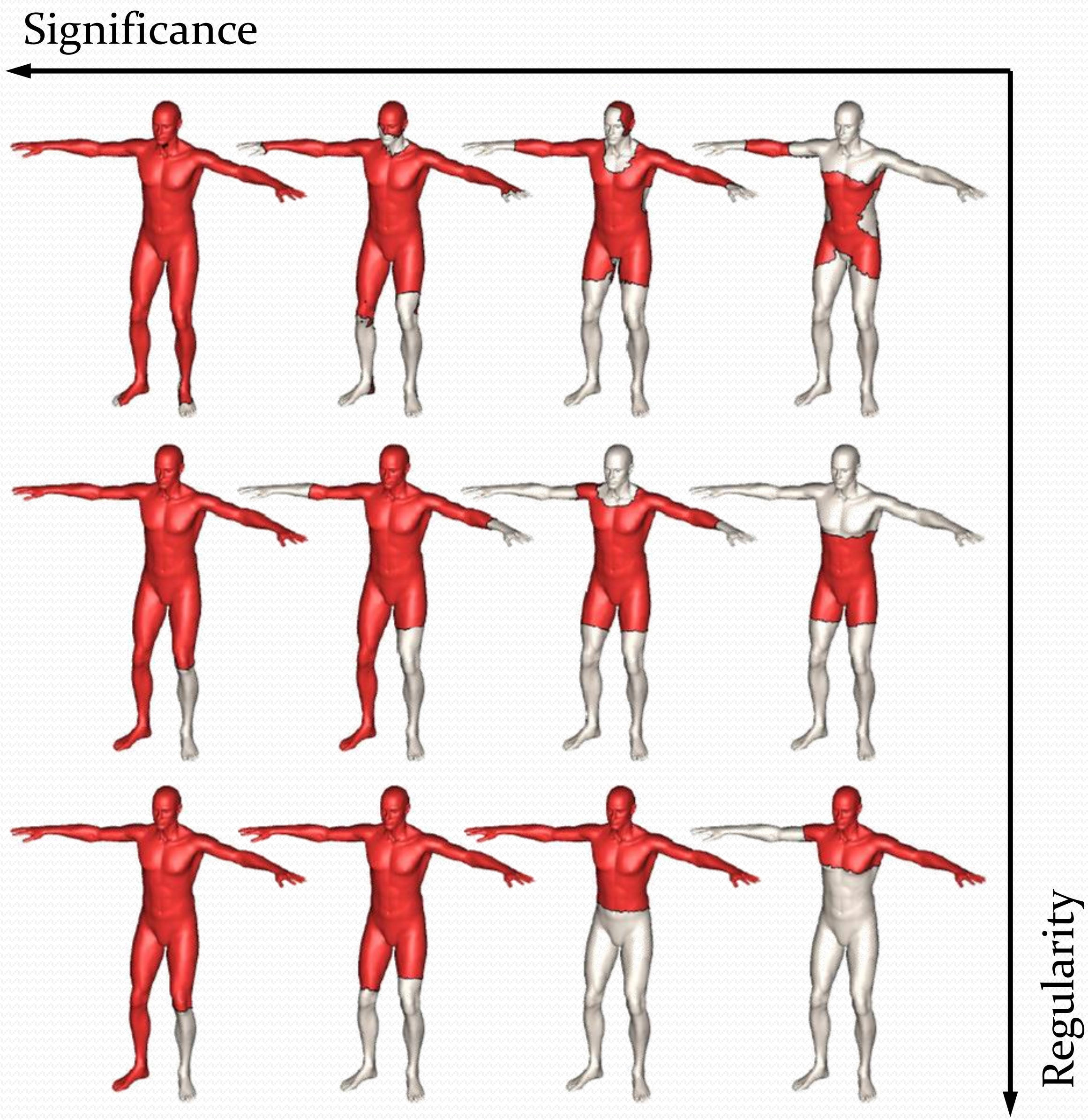
Irregular = long boundary | Regular = short boundary

Regularity will hopefully produce contiguous regions in our partial matching problem.

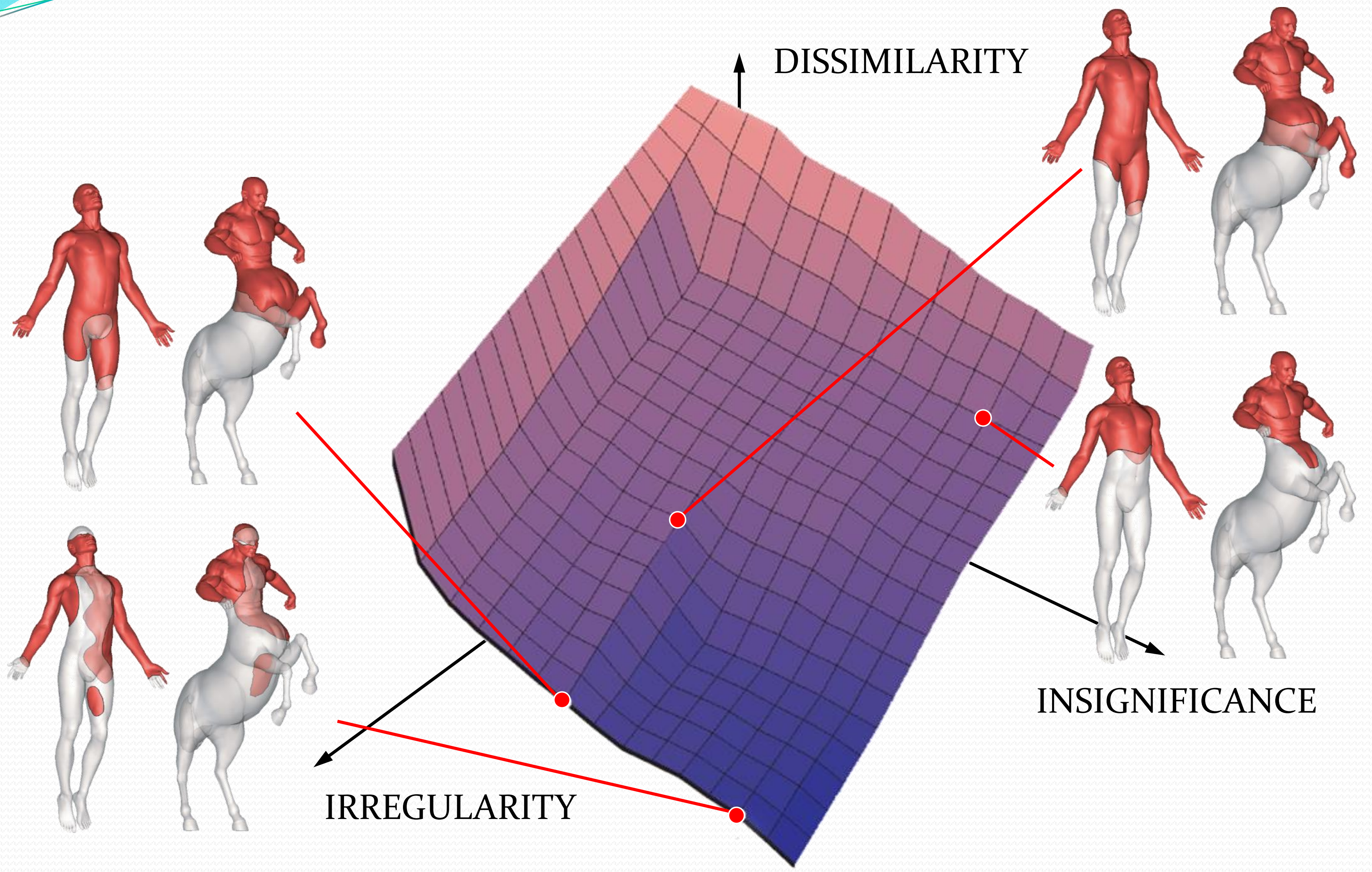
$$\min_{X' \subseteq X, Y' \subseteq Y} d(X', Y') - \alpha s(X', Y') + \beta \left(r(X') + r(Y') \right)$$

For this purpose, we can just use the **boundary length**:

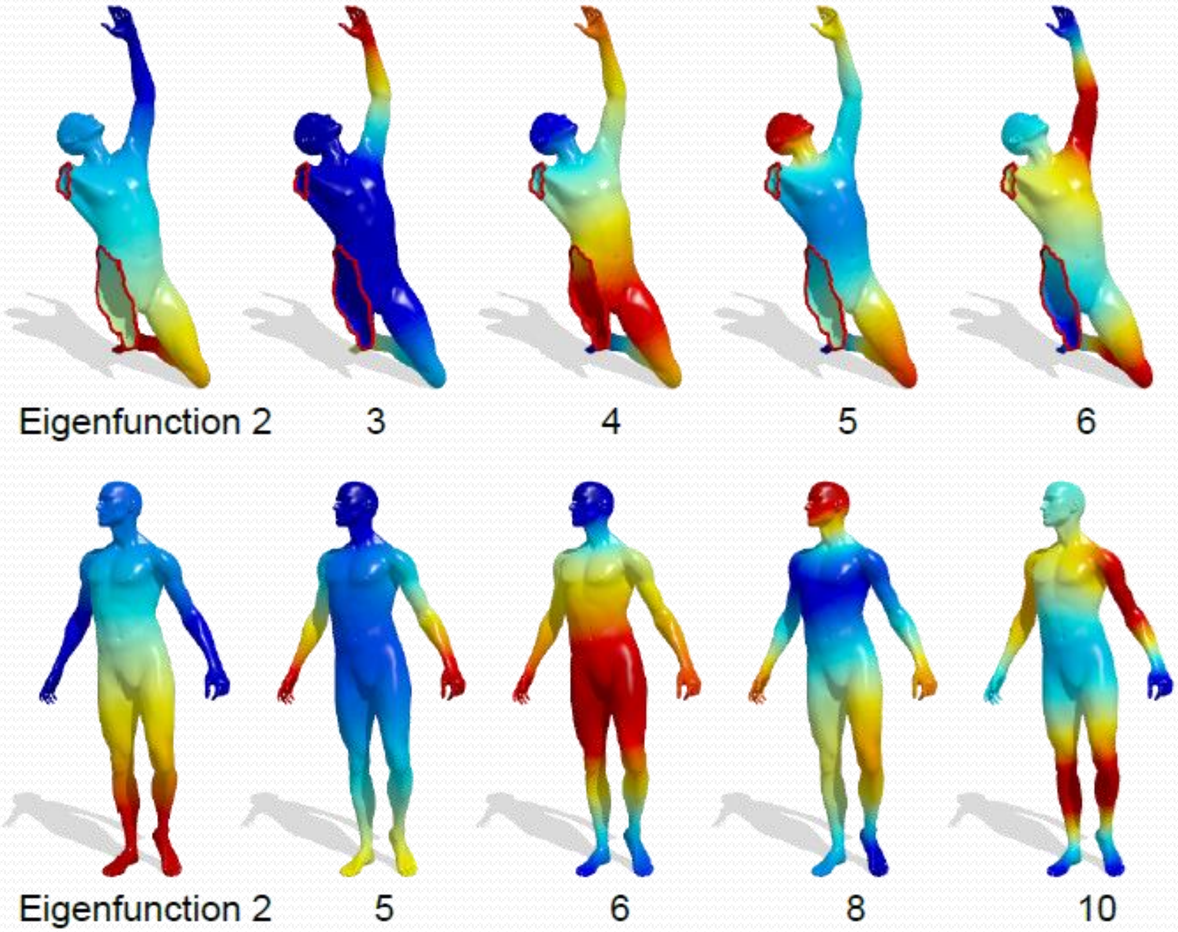
$$r(X') = \int_{\partial X'} d\ell \quad \overset{\text{fuzzy}}{\Longrightarrow} \quad r(u) = \int_X \|\nabla u(x)\| da(x)$$



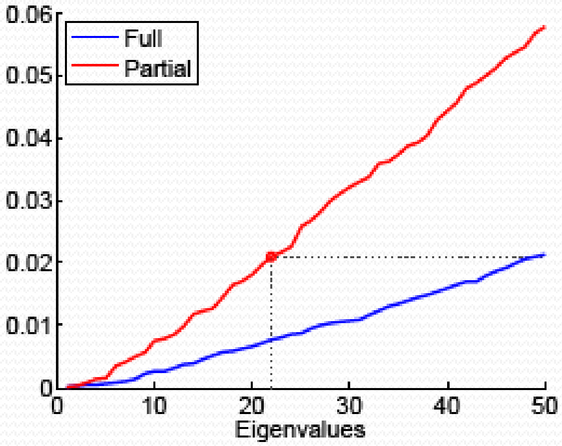
Significance
Regularity

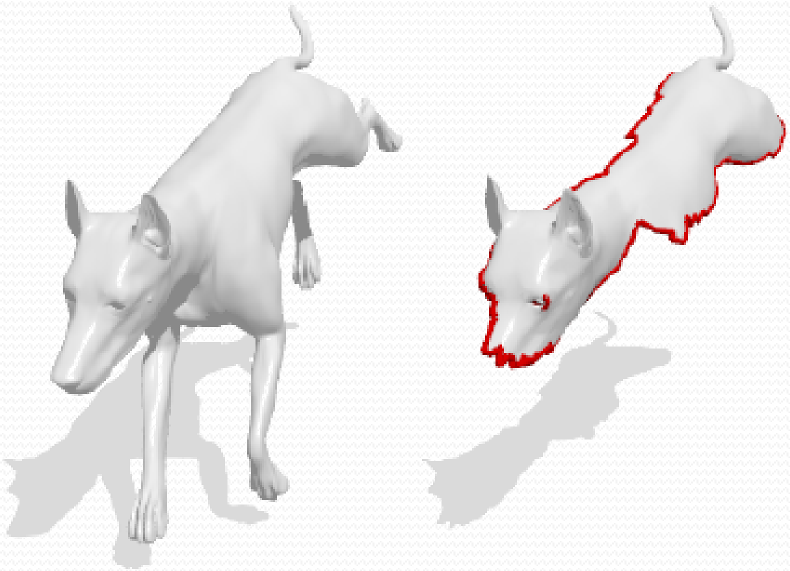

DISSIMILARITY
INSIGNIFICANCE
IRREGULARITY

LB eigenfunctions under partiality



LB eigenvalues under partiality

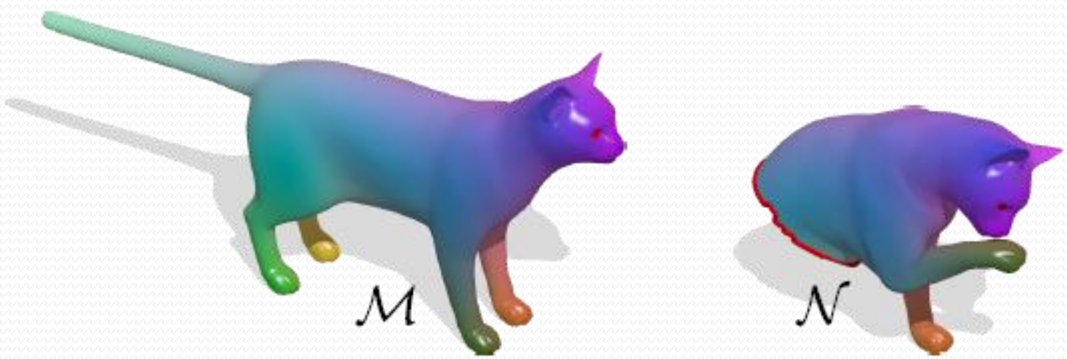




Recall from Weyl's law that the Laplacian spectrum grows linearly, and has an angular coefficient inversely proportional to the surface area:

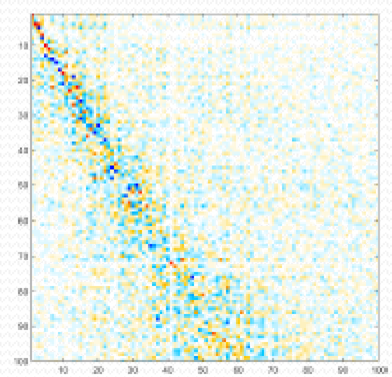
$$\lambda_j \sim \frac{\pi}{\int_S da} j$$

Partial functional map

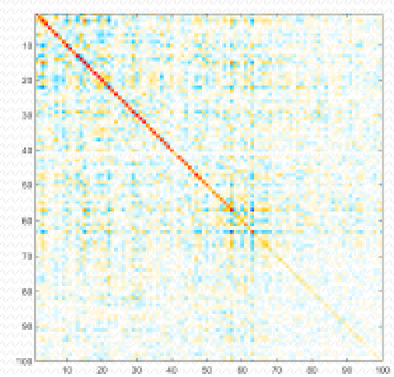


How do you expect the ground-truth functional map to look like, when represented as a matrix w.r.t. LB eigenbases?

The fact that some eigenfunctions are “skipped” in case of partiality, is manifested in a **C** with a slanted diagonal.



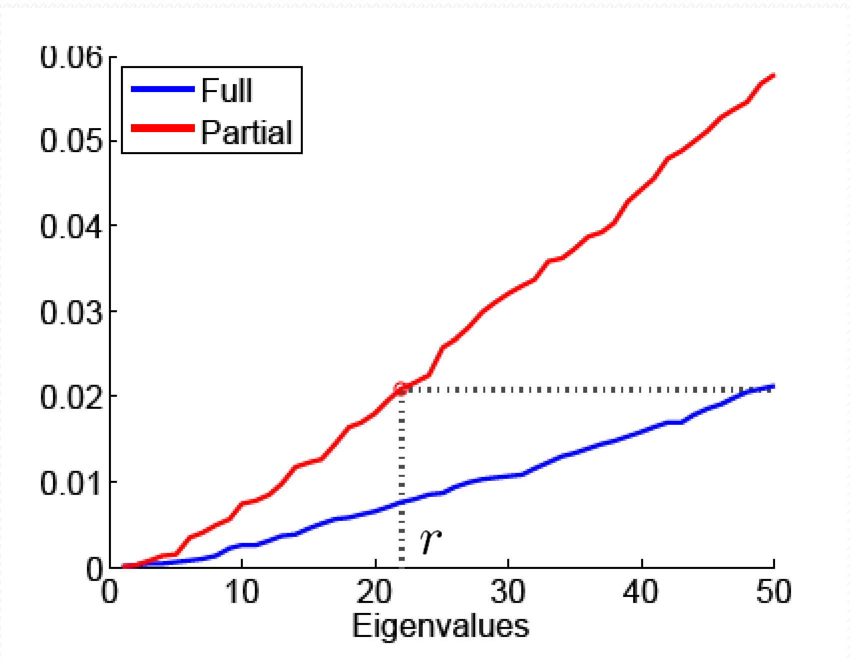
$\mathbf{C}$



$\mathbf{C}^\top\mathbf{C}$

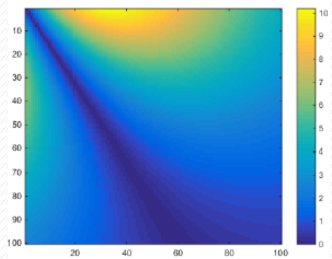
Estimating the slant

In fact, we can estimate the angle of the diagonal directly by looking at the eigenvalues of the two shapes (without even having to match them).



$$r = \max\{i \mid \lambda_i^{\mathcal{N}} < \max_j \lambda_j^{\mathcal{M}}\}$$

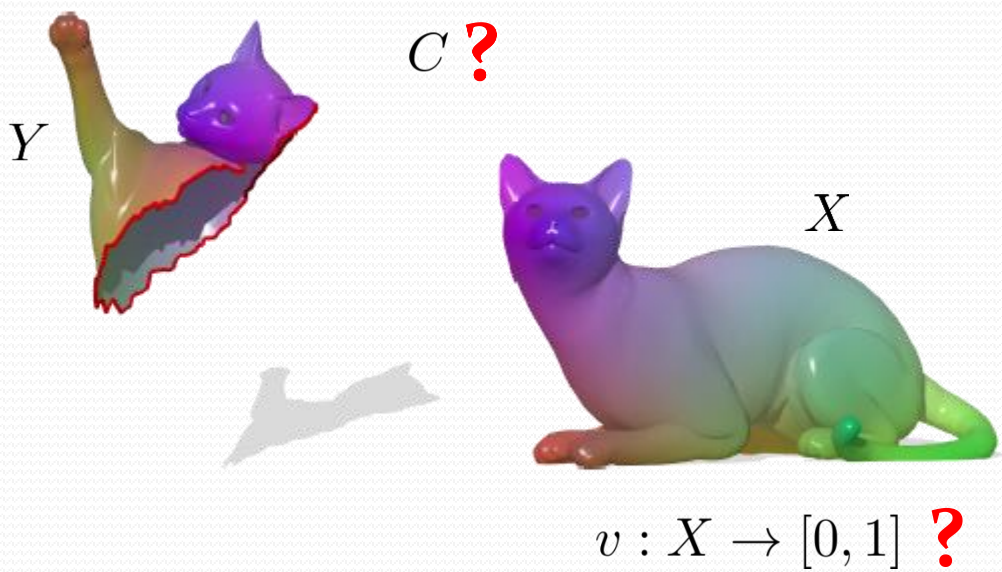
If matrix C is $k \times k$, then the slope of the diagonal is simply r/k



We can use this knowledge to construct a "mask" W for the unknown C.

Partial matching with func. maps

We now have all ingredients we need to formulate a partial matching problem with functional maps. For simplicity, we will only consider the case in which the first shape is partial, and the second one is full. Since the *matching region in the full shape is unknown*, we will have to solve for it as well. We will then use the notion of membership function to model it.



Partial matching with func. maps

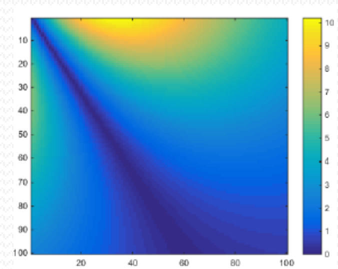
$$\min_{\substack{C \in \mathbb{R}^{k \times k} \\ v : X \to [0,1]}} \|C\Phi^{-1}F - \Psi^{-1} \begin{pmatrix} v_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_n \end{pmatrix} G\| + \rho_{\mathrm{corr}}(C) + \rho_{\mathrm{part}}(v)$$

this is just a reweighting of the data
according to the membership function *v*

$$\rho_{\mathrm{part}}(v) = (area(Y) - \int_X v(x) da(x)) + \int_X \|\nabla v(x)\| da(x)$$

significance + regularity

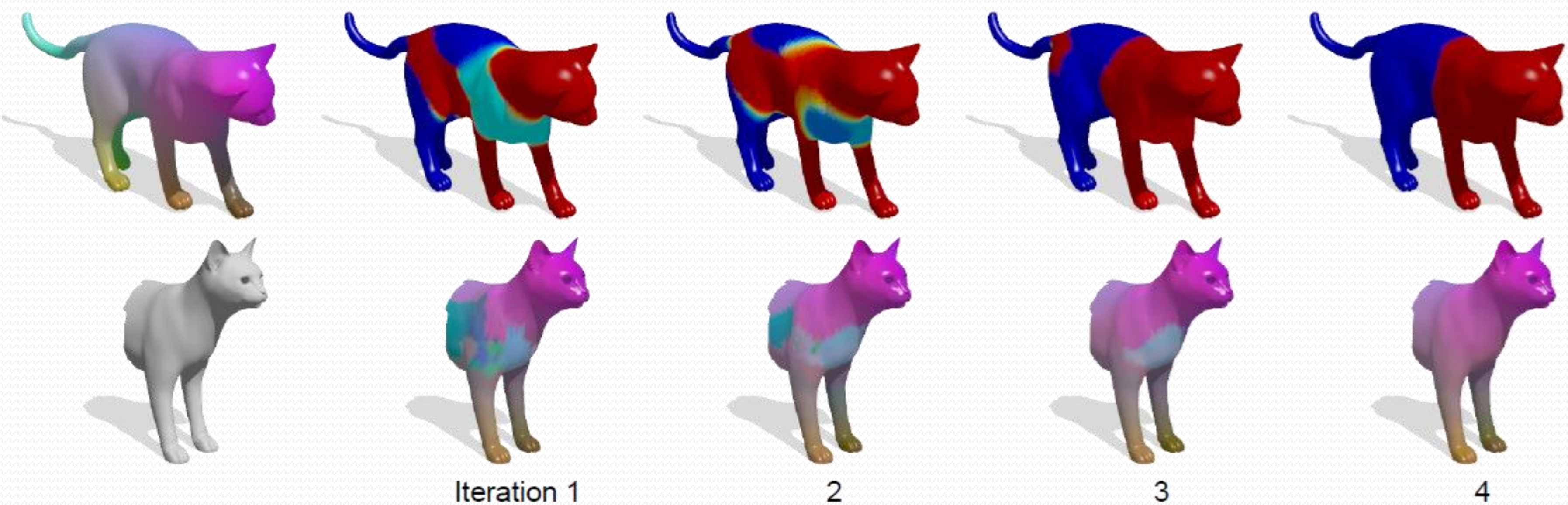
$$\rho_{\mathrm{corr}}(C) = \|C \circ W\|_F^2$$, where $W =$



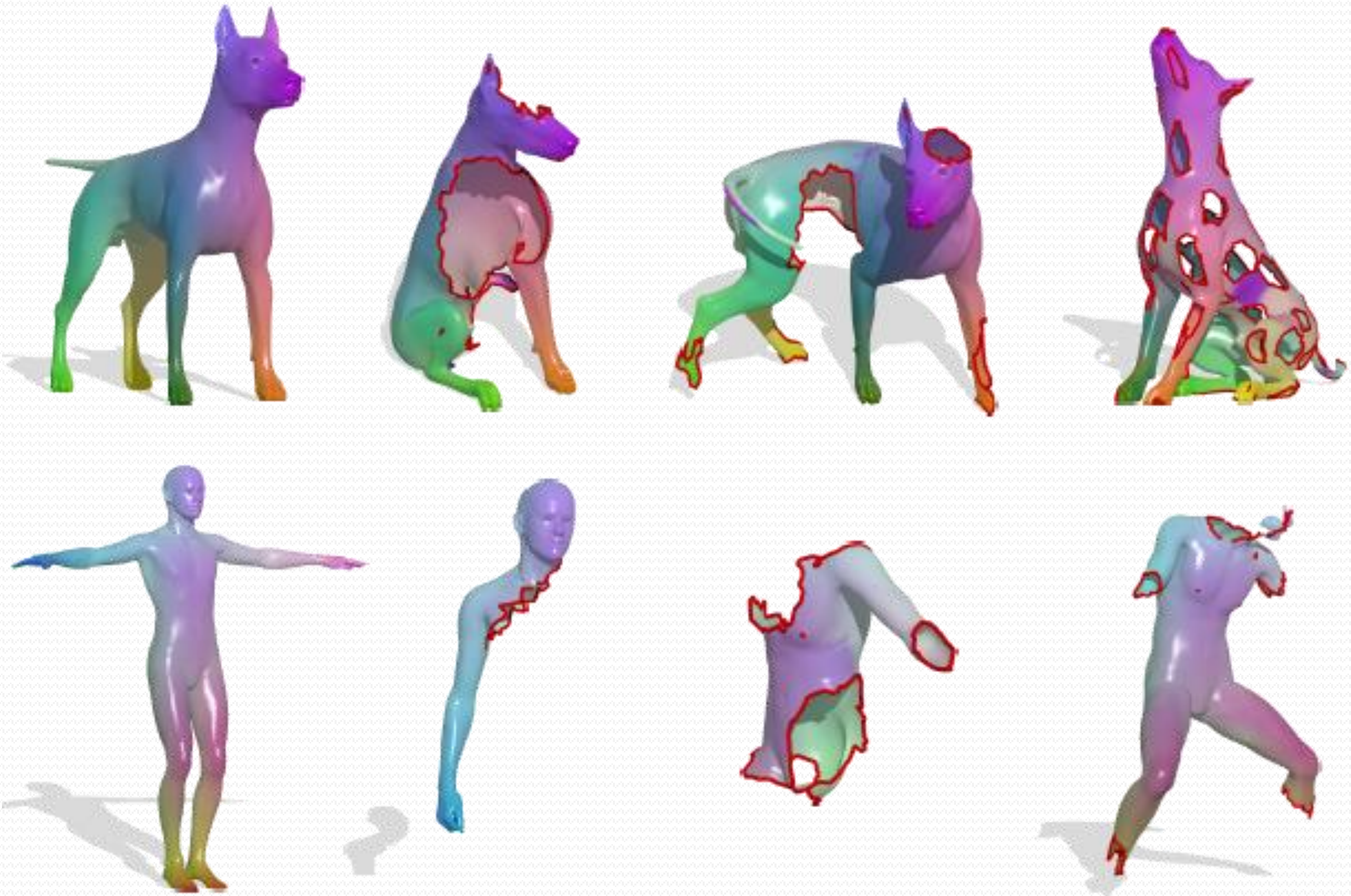
partial func. map

Optimization

Instead of optimizing over C and v simultaneously, we proceed just like before: we alternate between the two. We fist fix C and optimize for v, and then fix v and optimize for C. This is repeated until convergence.



Examples



Suggested reading

- *Partial similarity of objects, or how to compare a centaur to a horse*. Bronstein et al. IJCV 2009.
- *Not only size matters: Regularized partial matching of nonrigid shapes*. Bronstein and Bronstein. NORDIA 2008.
- *Partial functional correspondence*. Rodolà et al. 2015 http://arxiv.org/abs/1506.05274