Analysis of Three-Dimensional Shapes
(IN2238, TU München, Summer 2015)

The Laplacian
(18.05.2015)

Dr. Emanuele Rodolà
rodola@in.tum.de
Room 02.09.058, Informatik IX

Wrap-up

We defined the gradient as the unique ∇f such that:

$$\langle \nabla f, \vec{v} \rangle = df_p(\vec{v})$$

Interestingly, by passing to local coordinates, we noticed that we can compute the directional derivative directly in *U*, as:

$$df_p(\vec{v}) = (\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \text{where} \quad \tilde{f} = f \circ \mathbf{x}$$

defined on the surface *S*

defined on the parameter domain *U*

We can thus write $\langle \nabla f, \vec{v} \rangle = (\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Wrap-up

Using the bilinear definition of first fundamental form, we can also write

$$\langle \nabla f, \vec{v} \rangle = I_p(\nabla f, \vec{v}) = \begin{pmatrix} f_1 & f_2 \end{pmatrix} g \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Together with the last equation from the previous slide, we have

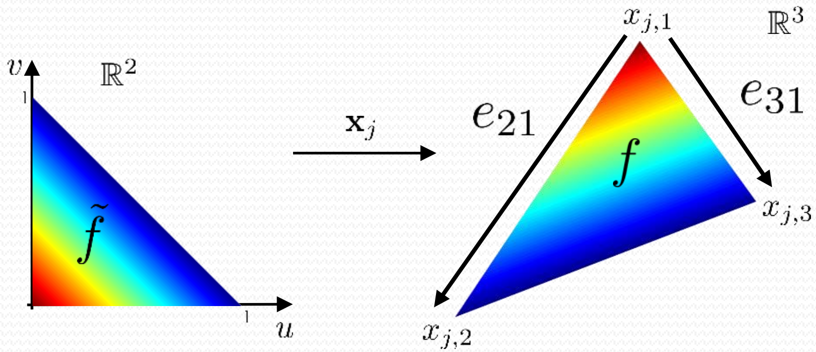
$$(\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \end{pmatrix} g \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

And thus we finally obtain:

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = g^{-1} \nabla \tilde{f}$$

Wrap-up

$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



$$\mathrm{D}\mathbf{x} = (\mathbf{x}_u, \mathbf{x}_v) = (e_{21}, e_{31})$$

$$g_j = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31}\rangle \\ \langle e_{21}, e_{31}\rangle & \|e_{31}\|^2 \end{pmatrix}$$

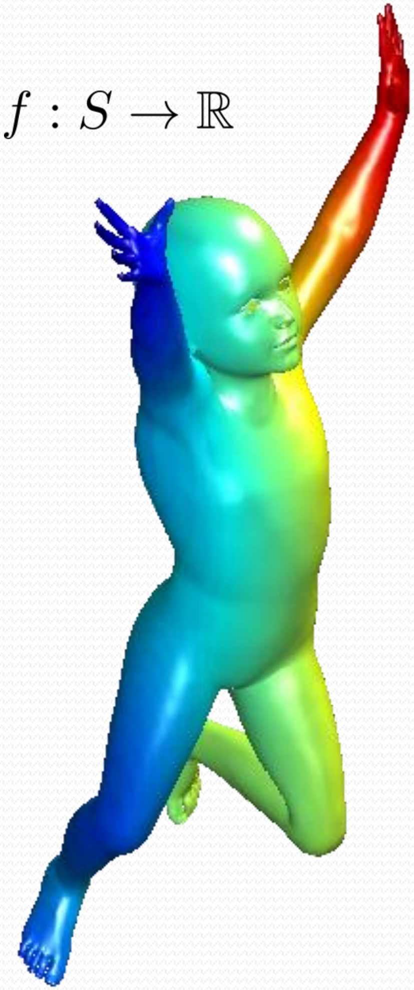
The discrete gradient is then given by:

$$\nabla f = \mathrm{D}\mathbf{x}\ g^{-1}\nabla\tilde{f} = (e_{21}, e_{31}) \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}^{-1} \begin{pmatrix} f(x_{j,2}) - f(x_{j,1}) \\ f(x_{j,3}) - f(x_{j,1}) \end{pmatrix}$$

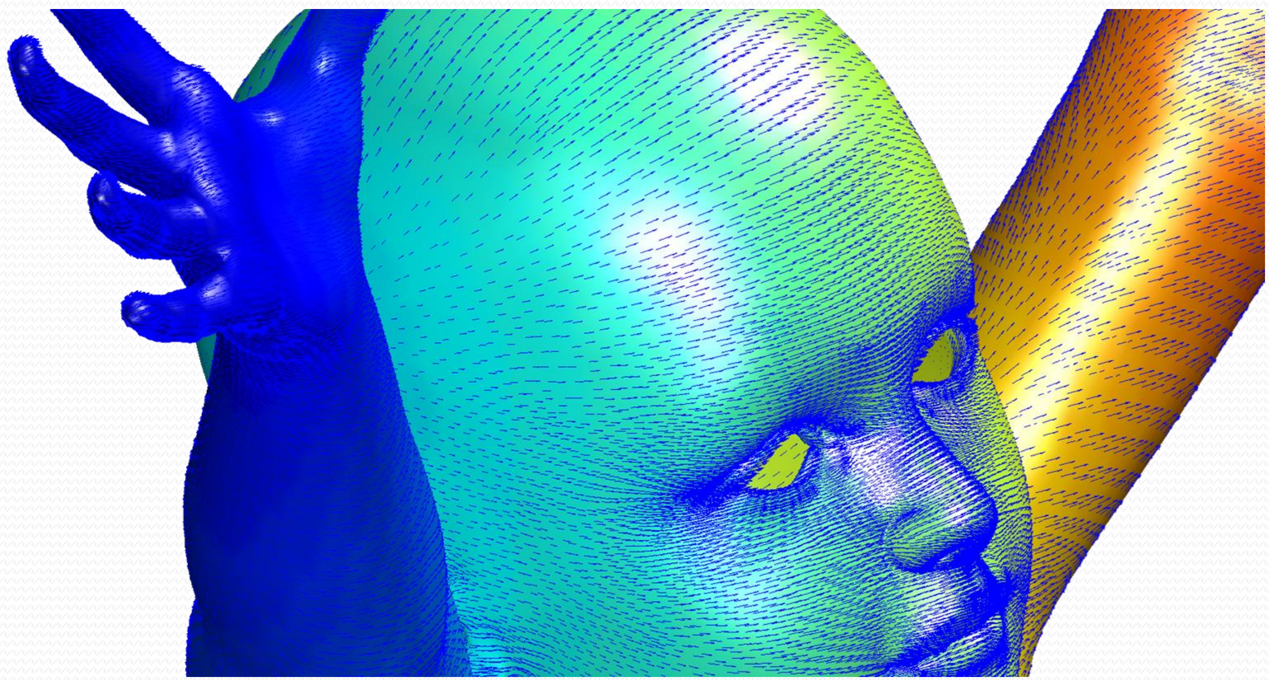
Also observe that: $\begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}^{-1} = \begin{pmatrix} G_j & -F_j \\ -F_j & E_j \end{pmatrix} \dfrac{1}{\det g_j}$

Wrap-up

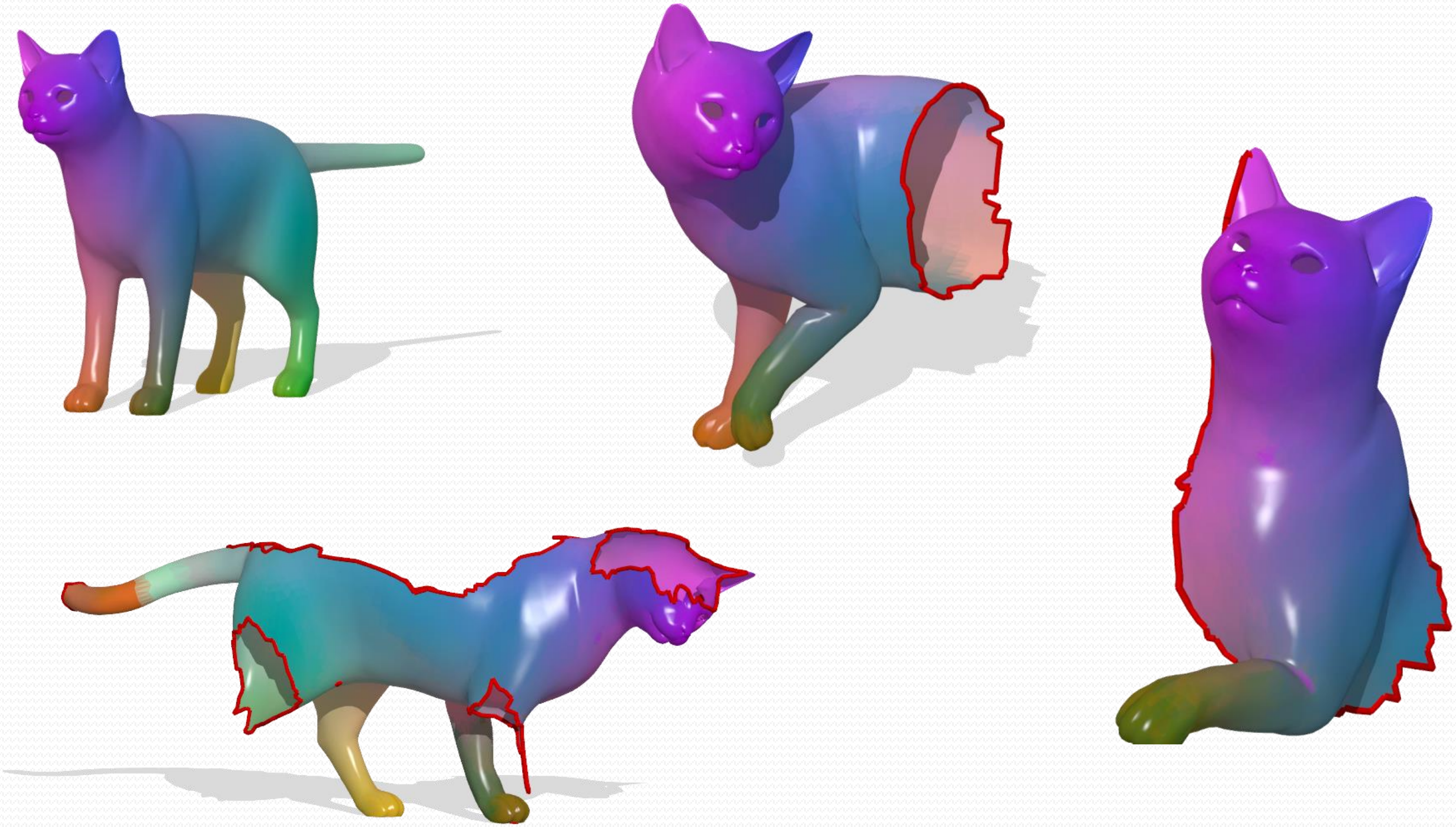
$f : S \to \mathbb{R}$



$\nabla f : S \to T_p S$



Shape matching



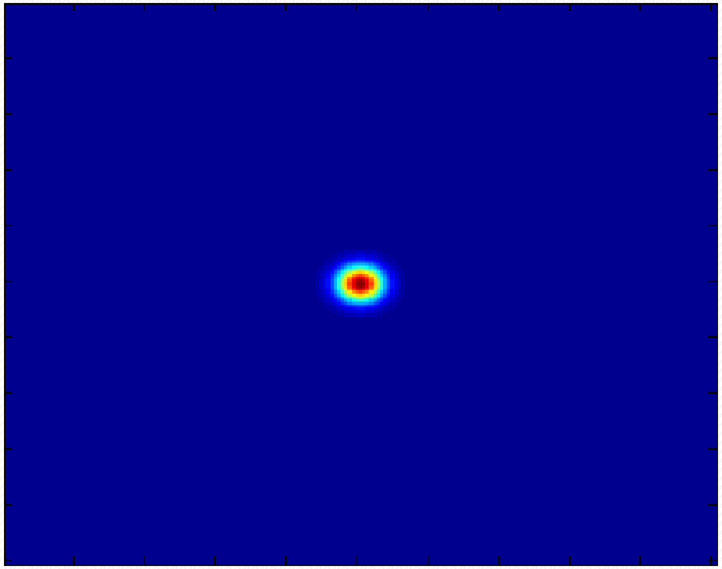
Heat diffusion in R^n

For an open subset $U \subset \mathbb{R}^n$ the diffusion of heat is described by the heat equation:

$$\frac{\partial \tilde{u}(x,t)}{\partial t} = \Delta \tilde{u}(x,t)$$

$$\tilde{u}(x,0) = \tilde{u}_0(x)$$

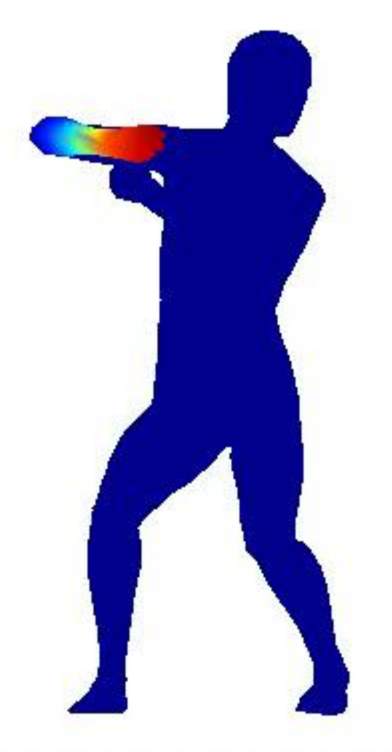
$$\tilde{u}(\partial U, t) = \ldots$$



$$\Delta \tilde{u}(x,t) = \mathrm{div}(\nabla \tilde{u}) = \sum_{i=1}^{n} \frac{\partial^2 \tilde{u}(x,t)}{\partial x_i^2}$$

Heat diffusion on surfaces

For a regular surface *S* the diffusion of heat is described by the heat equation:



$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t)$$

$$u(x,0) = u_0(x)$$

$$u(\partial U, t) = \ldots$$

$$\Delta u(x,t) = \mathrm{div}(\nabla u) = ?$$

How to compute the divergence of a vector field on a manifold?

Inner product on a manifold

One thing we did not define yet is the **inner product** of two scalar functions over a manifold. Given a regular surface *S* and functions $f, g : S \to \mathbb{R}$, we define their inner product as:

$$\langle f, g \rangle_S = \int_S f(x)g(x)dx$$

Applying our definition for integration on a manifold, we get to the expression in local coordinates:

$$\langle f, g \rangle_S = \int\int_U \tilde{f}(u,v)\tilde{g}(u,v)\sqrt{\det g} du dv$$

The divergence theorem in R^n

We will proceed similarly to how we did in the case of the gradient.

Namely, given a smooth function $\tilde{f} : U \to \mathbb{R}$ vanishing on the boundary ∂U, and a smooth vector field $\vec{\tilde{V}} = (\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_n) : U \to \mathbb{R}^n$, consider the inner product:

$$\langle \nabla \tilde{f}, \vec{\tilde{V}} \rangle_U = \int_U \langle \nabla \tilde{f}(x), \vec{\tilde{V}}(x) \rangle dx = \int_U \sum_i \frac{\partial \tilde{f}(x)}{\partial x_i} \tilde{V}_i(x) dx$$

Integration by parts in multiple dimensions

$$= - \int_U \sum_i \tilde{f}(x) \frac{\partial \tilde{V}_i(x)}{\partial x_i} dx + \int_{\partial U} \ldots$$

$$= - \int_U \tilde{f}(x) \underbrace{\sum_i \frac{\partial \tilde{V}_i(x)}{\partial x_i}}_{\mathrm{div}(\vec{\tilde{V}})} dx = -\langle \tilde{f}, \mathrm{div} \vec{\tilde{V}} \rangle_U$$

The divergence theorem

$$\langle \nabla f, \vec{V} \rangle = -\langle f, \mathrm{div} \vec{V} \rangle$$

Notice that the two inner products operate on different quantities, namely **vector fields** in the first case, and **functions** in the second case.

$\int_S \langle u, v \rangle$ $\qquad\qquad$ $\int_S fg$

In addition, formally the above relation characterizes the divergence as the **negative adjoint operator** to the gradient.

We will use this relation to define div *V* in local coordinates, since we already know what the gradient and the inner products mean on a Riemannian manifold.

Divergence on a surface

We use the divergence theorem to **define** the notion of divergence on a surface.

Let S be a regular surface *without boundary*, and let $V(p) = V_1\mathbf{x}_u + V_2\mathbf{x}_v \in T_pS$ be a smoothly changing vector field on S. We define $\operatorname{div} V : S \to \mathbb{R}$ to be the unique linear function satisfying the divergence theorem:

$$\int_S f(x)\underbrace{\operatorname{div} V(x)}\,dx = -\int_S \langle \nabla f(x), V(x)\rangle dx \qquad \forall f \in C^\infty(S)$$

we want to obtain an expression
for $\operatorname{div} V$ in local coordinates

Divergence in local coordinates

$$-\langle \nabla f, \vec{V} \rangle = \langle f, \mathrm{div}\vec{V} \rangle$$

Let us first rewrite the left-hand term in local coordinates:

$$-\langle \nabla f, \vec{V} \rangle = -\int_S \langle \nabla f(x), \vec{V}(x) \rangle dx \ = -\int_S I_p(\nabla f, \vec{V}) dx$$

$$= -\sum_j \int_{\tilde{U}_j} (g_j^{-1} \nabla \tilde{f}_j)^T g_j \vec{\tilde{V}}_j \sqrt{\det g_j}\, dudv \quad \leftarrow$$

No open subset of $\mathbb{R}^2$ is diffeomorphic to a regular surface without boundary, hence we sum over surface patches.

$$= -\sum_j \int_{\tilde{U}_j} (\nabla \tilde{f}_j)^T \vec{\tilde{V}}_j \sqrt{\det g_j}\, dudv$$

$$= -\sum_j \langle \nabla \tilde{f}_j, \sqrt{\det g_j} \vec{\tilde{V}}_j \rangle_{\tilde{U}_j}$$

Divergence in local coordinates

$$-\langle \nabla f, \vec{V} \rangle = \langle f, \mathrm{div}\vec{V} \rangle$$

Warning: even though we are considering **open** subsets of $\mathbb{R}^2$, the integration-by-parts rule operates **along their boundary**. However, it can be shown that the value of this integral is zero.

$$-\langle \nabla f, \vec{V} \rangle = -\sum_j \langle \nabla \tilde{f}_j, \sqrt{\det g_j} \tilde{\vec{V}}_j \rangle_{\tilde{U}_j}$$

Integration by parts (recall that we are in $\mathbb{R}^2$ now, see **slide #11**)

$$= \sum_j \langle \tilde{f}_j, \mathrm{div}(\sqrt{\det g_j} \tilde{\vec{V}}_j) \rangle_{\tilde{U}_j} - \cancel{\sum_j \int_{\partial \tilde{U}_j}} \cdots$$

$$= \int_U \tilde{f}(x) \sum_i \frac{\partial}{\partial x_i} (\sqrt{\det g} \tilde{\vec{V}}_i(x)) dx$$

In the last step we simply define:

$$U \equiv \bigcup \tilde{U}_j$$
$$\tilde{f}|_{\tilde{U}_j} \equiv \tilde{f}_j$$
$$\tilde{V}|_{\tilde{U}_j} \equiv \tilde{V}_j$$
$$\tilde{g}|_{\tilde{U}_j} \equiv \tilde{g}_j$$
$$dx \equiv du dv$$

Divergence in local coordinates

$$-\langle \nabla f, \vec{V} \rangle = \langle f, \mathrm{div}\vec{V} \rangle$$

$$-\langle \nabla f, \vec{V} \rangle = \int_U \tilde{f}(x) \sum_i \frac{\partial}{\partial x_i} (\sqrt{\det g}\tilde{\vec{V}}_i(x))dx$$

$$\langle f, \mathrm{div}\vec{V} \rangle = \int_U \tilde{f}(x) \mathrm{div}\tilde{\vec{V}}(x) \sqrt{\det g} dx$$

Since this equality has to hold for every f, we can deduce:

$$\mathrm{div}\vec{V}(x) = \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x_i} \left(\vec{V}_i(x) \sqrt{\det g} \right)$$

Laplacian in local coordinates

$$-\langle \nabla f, \vec{V} \rangle = \langle f, \mathrm{div} \vec{V} \rangle$$

If $\vec{V} = \nabla v$ for some differentiable function $v : S \to \mathbb{R}$ then we obtain

$$-\langle \nabla f, \nabla v \rangle = \langle f, \mathrm{div} \nabla v \rangle$$

The operator $\Delta := \mathrm{div} \circ \nabla$ is called the **Laplace-Beltrami operator**.

This relationship brings along a number of consequences, which we will look into with more detail tomorrow.

Laplacian in local coordinates

We can combine the expressions we have for gradient and divergence, and in turn obtain an expression in local coordinates for the Laplacian:

$$\Delta f = \mathrm{div} \nabla f = \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x_i} \left((\nabla f(x))_i \sqrt{\det g} \right)$$

$$= \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x_i} \left((g^{-1} \nabla \tilde{f}(x))_i \sqrt{\det g} \right)$$

$$= \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x_i} \left(\sum_j g^{ij} \frac{\partial \tilde{f}(x)}{\partial x_j} \sqrt{\det g} \right)$$

$$= \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(g^{ij} \frac{\partial \tilde{f}(x)}{\partial x_j} \sqrt{\det g} \right)$$

Laplacian in local coordinates

$$\Delta f = \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(g^{ij} \frac{\partial \tilde{f}(x)}{\partial x_j} \sqrt{\det g} \right)$$

We won't need to use this complicated formula, as we will discretize the Laplacian using a different procedure.

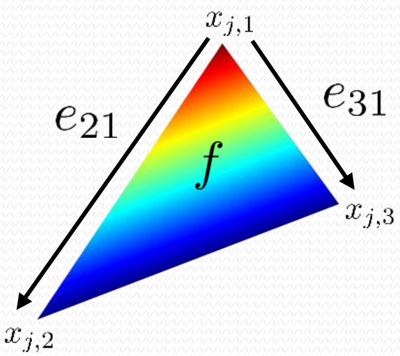
However, in this formula one sees that the Laplacian only involves the metric g.

This makes it an **isometry invariant**, and thus all other quantities directly derived from it will be isometry invariant as well (more on this tomorrow).

Discretization

We were able to discretize the gradient directly from its expression in local coordinates:

$$\nabla f = \mathrm{D}\mathbf{x}\ g^{-1}\nabla\tilde{f} = (e_{21}, e_{31}) \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}^{-1} \begin{pmatrix} f(x_{j,2}) - f(x_{j,1}) \\ f(x_{j,3}) - f(x_{j,1}) \end{pmatrix}$$



However, if we consider the expression we obtained for the divergence of the gradient $\vec{V} \equiv \nabla f$:

$$\mathrm{div}\vec{V}(x) = \frac{1}{\sqrt{\det g}} \sum_i \frac{\partial}{\partial x_i} \left(\vec{V}_i(x) \sqrt{\det g} \right)$$

We realize that we always get $\mathrm{div}\vec{V}(x) = 0$, since the vector field is constant within each triangle.

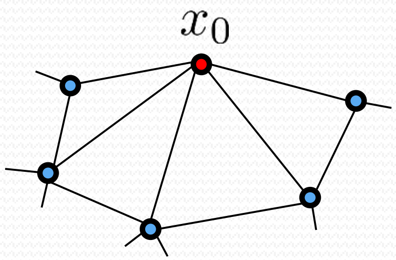
Action of the Laplacian

We will instead follow a different, **point-based** discretization.

To get an intuition of the action of the Laplacian, we can think of it as the generalization of the second derivative, as expressed by the relation:

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} = \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{2h^2}$$

Given a point x_0, the second derivative measures the difference of $f(x_0)$ to the average of f in an infinitesimal neighborhood.



In the discrete case, we consider the **1-ring neighborhood** of each vertex.

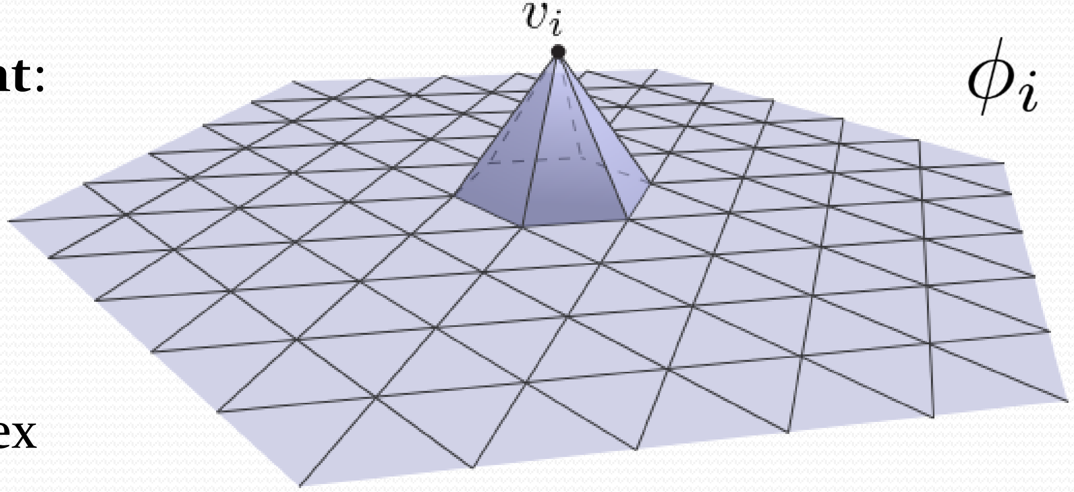
Finite elements (FEM)

By our usual assumption we are dealing with piecewise-linear functions, represented by the values they attain at each mesh vertex, that is:

$$f(v_i) = f_i$$

We can then express f in terms of a "hat" basis, where each hat function is called **finite element**:

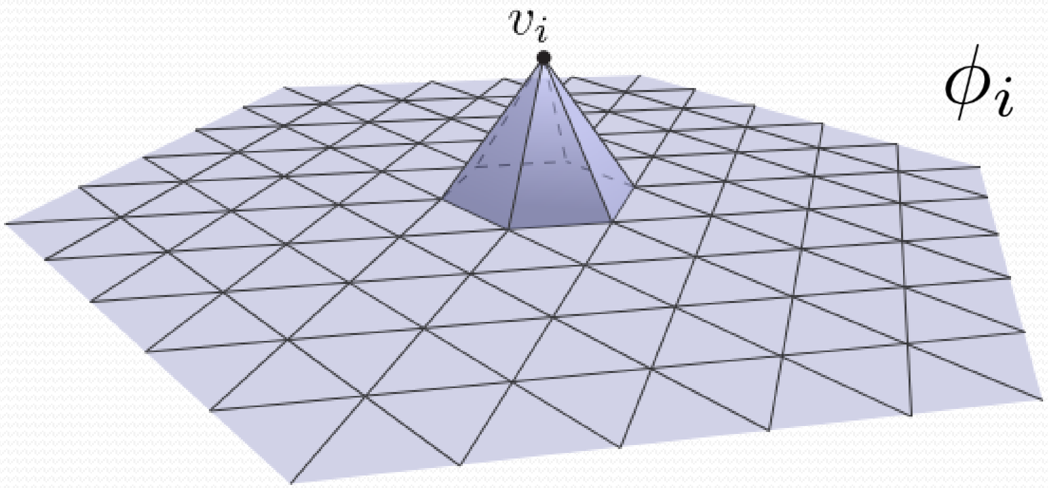
$$f(x) = \sum_i f_i \phi_i(x)$$



The finite element we consider **decreases linearly** from a vertex to its 1-ring neighbors.

Discrete Laplacian

$$f(x) = \sum_i f_i \phi_i(x)$$



The Laplacian of a function is a function itself, and thus can be expressed in the hat basis:

$$\Delta f(x) = h(x) = \sum_i h_i \phi_i(x)$$

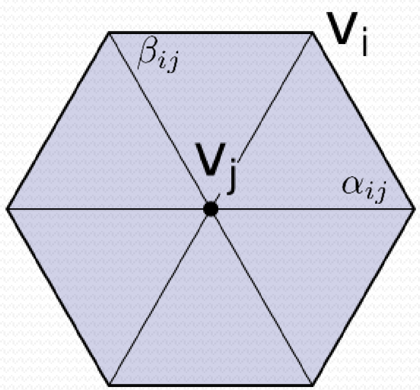
In matrix notation, we can write $Lf = h$. We are then interested in determining vector h whenever we are given f. Even better, it would be good to directly compute matrix L.

Weak formulation: 2nd term

Let's see what happens when we take inner products with a basis function ϕ_j , on both sides of $h = \Delta f$:

$$\langle h, \phi_j \rangle = \langle \Delta f, \phi_j \rangle$$

Let us first rewrite the second term:



$$\begin{aligned}\langle \Delta f, \phi_j \rangle &= -\langle \nabla f, \nabla \phi_j \rangle \\ &= -\langle \nabla \sum_i f_i \phi_i, \nabla \phi_j \rangle \\ &= -\sum_i f_i \underbrace{\langle \nabla \phi_i, \nabla \phi_j \rangle}_{-C_{ij}} \\ &= (Cf)_j\end{aligned}$$

$$(Cf)_j = \frac{1}{2} \sum_{i \in N(j)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_i - f_j)$$

Exercise! See also first suggested reading

The stiffness matrix

$$\langle \Delta f, \phi_j \rangle = (Cf)_j$$

From this relation it seems like we are done, since we could just go ahead and claim C to be the matrix representation of Δ.

If ϕ_j was an indicator vector in the usual sense, this would be true, since we would have $\langle \Delta f, \phi_j \rangle = (\Delta f)_j$.

However, in order to account for the mesh structure (*i.e.*, **local areas**), we chose *hat functions instead of sharp indicator functions*. In order to complete our derivation, we somehow have to incorporate areas in the discretization.

We call matrix C the **stiffness matrix**. Notice that it has the same structure as an adjacency matrix, with different weights.

Weak formulation: 1st term

$$\langle h, \phi_j \rangle = \langle \Delta f, \phi_j \rangle$$

Recall that we can write $h(x) = \sum_i h_i \phi_i(x)$. Then, the first term becomes:

$$\langle h, \phi_j \rangle = \sum_i h_i \langle \phi_i, \phi_j \rangle = \sum_i h_i \underbrace{\int \phi_i(x) \phi_j(x) dx}_{M_{ij}} = (Mh)_j$$

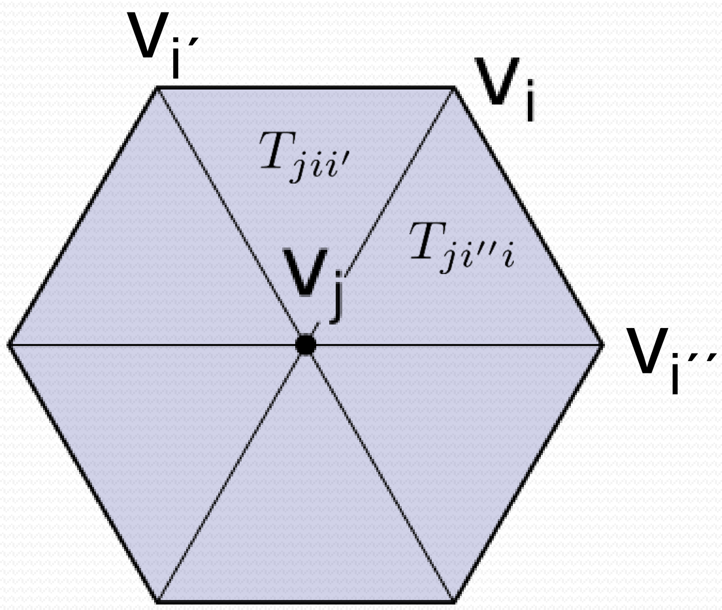
And then, together with the previous result we can write:

$$(Mh)_j = (Cf)_j \quad \forall j \quad \Rightarrow \quad Mh = Cf \quad \Rightarrow \quad \boxed{h = M^{-1} C f = L f}$$

The mass matrix

The integrals over the whole surface can be written as a sum of integrals over the triangles:

$$M_{ij} = \sum_{T \in S} \int_T \phi_i(x)\phi_j(x)dx$$

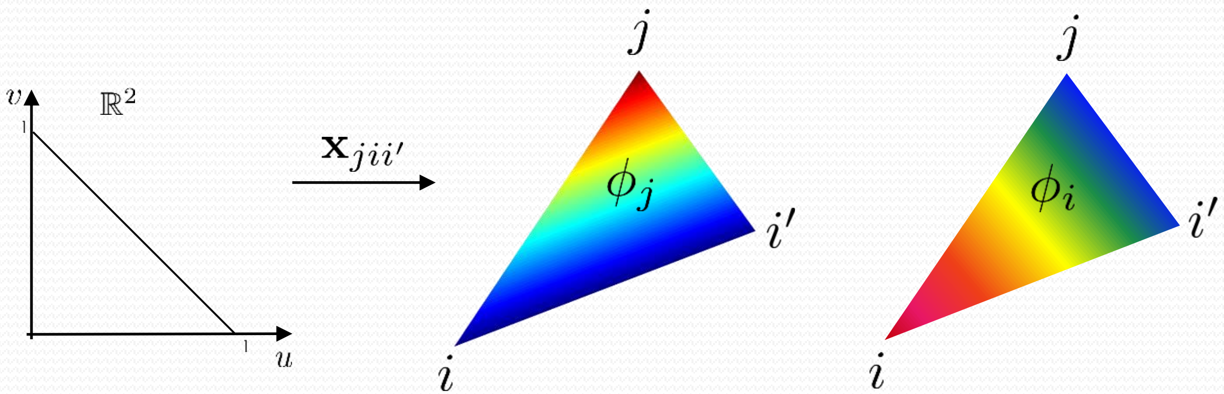


The integrand is zero whenever the edge $\{v_i, v_j\}$ is not an edge of triangle T. Thus, only two summands remain:

$$M_{ij} = \int_{T_{jii'}} \phi_i(x)\phi_j(x)dx + \int_{T_{ji''i}} \phi_i(x)\phi_j(x)dx$$

We will now calculate the first integral. The second one is similar.

The mass matrix



Functions ϕ_i, ϕ_j decrease linearly from 1 to 0.

Function $\mathbf{x}_{jii'}$ maps:

$(0,0) \mapsto i'$

$(0,1) \mapsto i$

$(1,0) \mapsto j$

Each function in the reference triangle is given by:

$$f(0,0)(1-u-v) + f(1,0)u + f(0,1)v$$

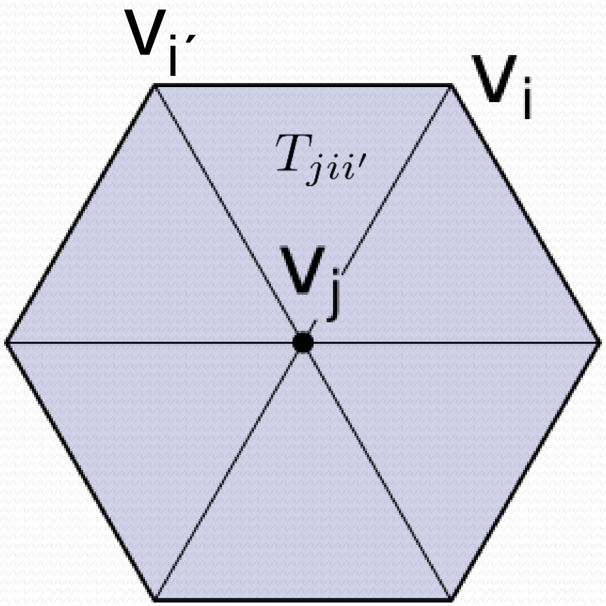
$$\int_{T_{jii'}} \phi_i(x)\phi_j(x)dx = 2A(T_{jii'}) \int_0^1 \int_0^{1-u} vu \; dvdu$$

$$= 2A(T_{jii'})\tfrac{1}{24} = \tfrac{1}{12}A(T_{jii'})$$

The mass matrix

We still have to compute the diagonal elements of the mass matrix. In this case, the support of the integrand covers all triangles in the neighborhood of the vertex:

$$M_{jj} = \sum_{T \in S} \int_T \phi_j(x)\phi_j(x)dx$$
$$= \sum_{i \in N(j)} \int_{T_{jii'}} \phi_j(x)\phi_j(x)dx$$



Similarly to before, when we discretize we get:

$$\int_{T_{jii'}} \phi_j(x)^2 dx = 2A(T_{jii'}) \int_0^1 \int_0^{1-u} u^2 \, dvdu$$
$$= \frac{1}{6} A(T_{jii'})$$

Discrete Laplacian: wrap-up

We discretized the Laplace-Beltrami operator using FEM.
The method starts from choosing a finite element basis $\{\phi_j\}_{j=1,\ldots,n}$, and writing the weak relation:

$$\langle h, \phi_j \rangle = \langle \Delta f, \phi_j \rangle$$

The two sides of the equation can be rewritten as $Mh = Cf$, hence giving us:

$$L = M^{-1} C$$

The mass matrix M and stiffness matrix C are sparse, $n \times n$ matrices that can be easily computed for any given mesh.

Discrete Laplacian: wrap-up

The stiffness and mass matrices are respectively given by:

$$C_{ij} = \begin{cases} \dfrac{\cot\alpha_{ij} + \cot\beta_{ij}}{2} & (i,\, j)\ \text{edge} \\ -\sum\limits_{k \in N(i)} C_{ik} & i = j \end{cases}$$

$$M_{ij} = \begin{cases} \dfrac{A(T_1) + A(T_2)}{12} & (i,\, j)\ \text{edge} \\ \dfrac{\sum_{k \in N(i)} A(T_k)}{6} & i = j \end{cases}$$

T1, T2 are the triangles that share the edge (i,j)

A “lumped” variant

A variant of the FEM discretization is much more often see in the literature.

In particular, the mass matrix is often approximated as the diagonal matrix with elements:

$$M_{ii} = \sum_{k \in N(i)} \frac{A(T_k)}{3}$$

This discretization is called the “lumped” Laplacian.

Notice that, although an approximation to the correct FEM approach, the lumped mass matrix is easier to invert and to deal with.

Suggested reading

- http://brickisland.net/cs177/?p=309
- http://wwwmath.uni-muenster.de/num/Vorlesungen/WissenschaftlichesRechnen_WS1213/Dateien/Fem-intro.pdf
- "*Linear Algebra Done Right – Second Edition*". S. Axler. Pages 117-121
- "*Discrete Laplace-Beltrami operators for shape analysis and segmentation*". M. Reuter et al, CAG 33, 2009. Sections 1 to 2.1.2