Multiple View Geometry: Exercise Sheet 1
Solution of the theoretical exercises

1.

$$\begin{aligned}
\lambda_a \cdot \langle v_a, v_b \rangle &= \langle \lambda_a v_a, v_b \rangle \\
&= \langle A v_a, v_b \rangle \\
&= v_a^T A^T v_b \\
&= \langle v_a, A^T v_b \rangle \\
&\overset{(A=A^T)}{=} \langle v_a, A v_b \rangle \\
&= \lambda_b \cdot \langle v_a, v_b \rangle
\end{aligned}$$

$$\Rightarrow \lambda_a = \lambda_b \text{ if } \langle v_a, v_b \rangle \neq 0$$

2. $v_1, \ldots, v_n$ ONB of A with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$.
   $x = \sum_{i=1}^{n} \alpha_i v_i$ with $\alpha_i \in \mathbb{R}$, $||x||^2 = \sum_{i,j} \alpha_i \underbrace{v_i^T v_j}_{=\delta(i,j)} \alpha_j = \sum_i \alpha_i^2 \overset{!}{=} 1$

$$\begin{aligned}
x^T A x &= (\sum_{i=1}^{n} \alpha_i v_i)^T (\sum_{i=1}^{n} \alpha_i A v_i) \\
&= (\sum_{i=1}^{n} \alpha_i v_i)^T (\sum_{i=1}^{n} \alpha_i \lambda_i v_i) \\
&= \sum_{i=1}^{n} \underbrace{\alpha_i^2}_{:=\beta_i} \lambda_i
\end{aligned}$$

   with $\beta_i \geq 0$ and $\sum_i \beta_i = 1$. The term is minimized if $\beta_i = 0$ for $\lambda_i > \lambda_n$.

   If $\lambda_{n-1} \gneq \lambda_n$ there exist two solutions (we are interested in the α_i). Otherwise there are infinitely many solutions.

   The term is maximized if $\beta_i = 0$ for $\lambda_i < \lambda_1$.

3. (a) i. Closure:

$$L_1 = \begin{pmatrix} A_1 & b_1 \\ 0 & 1 \end{pmatrix} \in A(n),\ L_2 = \begin{pmatrix} A_2 & b_2 \\ 0 & 1 \end{pmatrix} \in A(n)$$
$$\Rightarrow L_1 L_2 = \begin{pmatrix} A_1 A_2 & A_1 b_2 + b_1 \\ 0 & 1 \end{pmatrix} \in A(n).$$

   ii. The associative law is generelly fulfilled for matrix multiplication.

   iii. Identity element: $\begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} \in A(n)$

   iv. Inverse element: $L = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in A(n) \rightarrow L^{-1} = \begin{pmatrix} A^{-1} & -A^{-1} b \\ 0 & 1 \end{pmatrix} \in A(n)$

(b) i. Closure:

$$L_1 = \begin{pmatrix} R_1 & b_1 \\ 0 & 1 \end{pmatrix} \in E(n),\ L_2 = \begin{pmatrix} R_2 & b_2 \\ 0 & 1 \end{pmatrix} \in E(n)$$
$$\Rightarrow L_1 L_2 = \begin{pmatrix} R_1 R_2 & R_1 b_2 + b_1 \\ 0 & 1 \end{pmatrix} \in E(n),$$

because: $R_1, R_2 \in O(n) : (R_1 R_2)^T (R_1 R_2) = R_2^T \underbrace{R_1^T R_1}_{=I} R_2 = R_2^T R_2 = I.$

ii. Identity element: $\begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} \in E(n)$

iii. Inverse element: $L = \begin{pmatrix} R & b \\ 0 & 1 \end{pmatrix} \in E(n) \to L^{-1} = \begin{pmatrix} R^{-1} & -R^{-1}b \\ 0 & 1 \end{pmatrix} \in E(n),$
because $R \in O(n) : (R^{-1})^T R^{-1} = (RR^T)^{-1} = (R^T R)^{-T} = I.$

(c) i. Closure:

$$L_1 = \begin{pmatrix} R_1 & b_1 \\ 0 & 1 \end{pmatrix} \in SE(n),\ L_2 = \begin{pmatrix} R_2 & b_2 \\ 0 & 1 \end{pmatrix} \in SE(n)$$
$$\Rightarrow L_1 L_2 = \begin{pmatrix} R_1 R_2 & R_1 b_2 + b_1 \\ 0 & 1 \end{pmatrix} \in SE(n),$$

because $R_1, R_2 \in SO(n) : \det(R_1 R_2) = \underbrace{\det(R_1)}_{=1} \cdot \underbrace{\det(R_2)}_{=1} = 1.$

ii. Identity element: $\begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} \in SE(n)$ because $I \in SO(n)$

iii. Inverse element: $L = \begin{pmatrix} R & b \\ 0 & 1 \end{pmatrix} \in SE(n) \to L^{-1} = \begin{pmatrix} R^{-1} & -R^{-1}b \\ 0 & 1 \end{pmatrix} \in SE(n),$
because $R \in SO(n) : \det(R^{-1}) = \dfrac{1}{\det(R)} = 1.$