THE HODGE CONJECTURE FOR SELF-PRODUCTS OF CERTAIN K3 SURFACES

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Abstract. We use a result of van Geemen [vG4] to determine the endomorphism algebra of the Kuga–Satake variety of a K3 surface with real multiplication. This is applied to prove the Hodge conjecture for self-products of double covers of $\mathbb{P}^2$ which are ramified along six lines.

1. Introduction

Let S be a complex K3 surface, i.e. a smooth, projective surface over $\mathbb{C}$ satisfying $H^1(S, \mathcal{O}_S) = 0$ and $\omega_S \simeq \mathcal{O}_S$. Let $T(S) \subset H^2(S, \mathbb{Q})$ be the rational transcendental lattice of S, defined as the orthogonal complement of the Néron–Severi group with respect to the intersection form. The algebra $E_S := \mathrm{End}_{\mathrm{Hdg}}(T(S))$ of endomorphisms of $T(S)$ which preserve the Hodge decomposition can be interpreted as a subspace of the space of (2,2)-classes on the self-product $S \times S$. The Hodge conjecture for $S \times S$ predicts that E_S consists of linear combinations of fundamental classes of algebraic surfaces in $S \times S$. Using the Lefschetz theorem on (1,1)-classes, it is easily seen that conversely the Hodge conjecture for $S \times S$ holds if E_S is generated by algebraic classes.

Mukai [Mu1] used his theory of moduli spaces of sheaves to prove that if the Picard number of S is at least 11, then any $\varphi \in E_S$ which preserves the intersection form on $T(S)$ can be represented as a linear combination of fundamental classes of algebraic cycles. Later this result was improved by Nikulin [N] on the base of lattice-theoretic arguments to the case that the Picard number of S is at least 5. In [Mu2], Mukai announced that using the theory of moduli spaces of twisted sheaves, the hypothesis on the Picard number could be omitted.

But how many isometries do exist in the algebra E_S? Results of Zarhin [Z] imply that E_S is an algebraic number field, which is either totally real (we say that S has *real multiplication*) or a CM field (S has *complex multiplication*). Isometries of $T(S)$ correspond to elements of norm 1 in E_S. If S has complex multiplication, one can use the fact that CM fields are generated as $\mathbb{Q}$-vector spaces by elements of norm 1 to see that E_S is generated by isometries. In combination with Mukai's results, this proves the Hodge conjecture for self-products of K3 surfaces with complex multiplication and with Picard number at least 5. This was noticed by Ramón-Marí [RM]. If S has real multiplication, the only Hodge isometries in E_S are plus or minus the

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identity. Thus, Mukai's results are no longer sufficient to prove the algebraicity of interesting classes in E_S.

In order to approach the case of real multiplication one passes from K3 surfaces to Abelian varieties by associating to a K3 surface S its Kuga–Satake Abelian variety A. By construction, see [KS], there exists an inclusion of Hodge structures $T(S) \subset H^2(A \times A, \mathbb{Q})$. Van Geemen [vG4] studied the Kuga–Satake variety of a K3 surface with real multiplication. He discovered that the corestriction of a certain Clifford algebra over E_S plays an important role for the Kuga–Satake variety of S. We rephrase and slightly improve his result which then reads as follows:

Theorem 1. *Let S be a K3 surface with real multiplication by a totally real number field E_S of degree d over $\mathbb{Q}$. Let A be a Kuga–Satake variety of S.*

Then there exists an Abelian variety B such that A is isogenous to $B^{2^{d-1}}$. The endomorphism algebra of B is

$$\mathrm{End}_{\mathbb{Q}}(B) = \mathrm{Cores}_{E/\mathbb{Q}}\, C^0(Q).$$

Here, $Q : T \times T \to E_S$ is a quadratic form on T which already appeared in Zarhin's paper [Z] and which will be reintroduced in Section 2.4, $C^0(Q)$ is the even Clifford algebra of Q over E_S and $\mathrm{Cores}_{E/\mathbb{Q}}\, C^0(Q)$ is the corestriction of this algebra. The corestriction of algebras will be reviewed in Section 3.2.

Theorem 1 leads to a better understanding of the phenomenon of real multiplication for K3 surfaces by allowing us to calculate the endomorphism algebra of the corresponding Kuga–Satake varieties. However, since the Kuga–Satake construction is purely Hodge-theoretic, this still gives no geometric explanation. Therefore, we focus on one of the few families of K3 surfaces for which the Kuga–Satake correspondence has been understood geometrically. This is the family of double covers of $\mathbb{P}^2$ ramified along six lines. Paranjape [P] found an explicit cycle on $S \times A \times A$ which realizes the inclusion of Hodge structures $T(S) \subset H^2(A \times A, \mathbb{Q})$. Building on the decomposition result for Kuga–Satake varieties we derive

Theorem 2. *Let S be a K3 surface which is a double cover of $\mathbb{P}^2$ ramified along six lines. Then the Hodge conjecture is true for $S \times S$.*

As pointed out by van Geemen [vG4], there are one-dimensional sub-families of the family of such double covers with real multiplication by a totally real quadratic number field. In conjunction with our Theorem 2, this allows us to produce examples of K3 surfaces S with non-trivial real multiplication for which $\mathrm{End}_{\mathrm{Hdg}}(T(S))$ is generated by algebraic classes. We could not find examples of this type in the existing literature.

The *plan of the paper* is as follows: In Section 2 we review Zarhin's results on the endomorphism algebra and on the special Mumford–Tate group of an irreducible Hodge structure of K3 type. Also, we recall from [vG4] how a Hodge structure of K3 type with real multiplication splits over a finite extension of $\mathbb{Q}$.

Section 3 is devoted to the proof of Theorem 1. After reviewing the definition of the corestriction of algebras, we explain in detail how the Galois group of a normal closure of E_S acts on the Kuga–Satake Hodge structure. This is the key of the proof.

In the final Section 4 we study double covers of $\mathbb{P}^2$ ramified along six lines. We review results of Lombardo [Lo] on the Kuga–Satake variety of such K3 surfaces, of Schoen [S] and van Geemen [vG2] on the Hodge conjecture for certain Abelian varieties of Weil type and of course Paranjape's [P] result on the algebraicity of the Kuga–Satake correspondence. Together with Theorem 1, they lead to the proof of Theorem 2.

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2. Hodge structures of K3 type with real multiplication

2.1. **Hodge structures of K3 type and their endomorphisms.** Let U(1) be the one-dimensional unitary group which is a real algebraic group. To fix notations we recall that a Hodge structure of weight k is a finite-dimensional $\mathbb{Q}$-vector space T together with a morphism of real algebraic groups $h : \mathrm{U}(1) \to \mathrm{GL}(T)_{\mathbb{R}}$ such that for $z \in \mathrm{U}(1)(\mathbb{R}) \subset \mathbb{C}$ the $\mathbb{C}$-linear extension of the endomorphism $h(z)$ is diagonalizable with eigenvalues $z^p\overline{z}^q$ where $p + q = k$ and $p, q \geq 0$ (cf. e.g. [vG4, 1.1]). The eigenspace to $z^p\overline{z}^q$ is denoted by $T^{p,q} \subset T_{\mathbb{C}}$.

A polarization of a weight k Hodge structure (T, h) is a bilinear form $q : T \times T \to \mathbb{Q}$ which is U(1)-invariant and which has the property that $(-1)^{k(k-1)/2}q(*, h(i)*) : T_{\mathbb{R}} \times T_{\mathbb{R}} \to T_{\mathbb{R}}$ is a symmetric, positive definite bilinear form.

Definition 2.1.1. A *Hodge structure of K3 type* (T, h, q) consists of a $\mathbb{Q}$-Hodge structure $(T, h : \mathrm{U}(1) \to \mathrm{GL}(T)_{\mathbb{R}})$ of weight 2 with $\dim_{\mathbb{C}} T^{2,0} = 1$ together with a polarization $q : T \times T \to \mathbb{Q}$.

Examples. The second primitive (rational) cohomology and the (rational) transcendental lattice of a projective K3 surface yield examples of Hodge structures of K3 type. More generally, the second primitive cohomology and the Beauville–Bogomolov orthogonal complement of the Néron–Severi group of an irreducible symplectic variety are Hodge structures of K3 type [GHJ, Part III].

Consider the Hodge decomposition

$$T_{\mathbb{C}} := T \otimes_{\mathbb{Q}} \mathbb{C} = T^{2,0} \oplus T^{1,1} \oplus T^{0,2}.$$

Since the quadratic form q is a polarization, this decomposition is q-orthogonal. Moreover, q is positive definite on $(T^{2,0} \oplus T^{0,2}) \cap T_{\mathbb{R}}$ and negative definite on $T^{1,1} \cap T_{\mathbb{R}}$.

Assume that T is an irreducible Hodge structure. Let $E := \mathrm{End}_{\mathrm{Hdg}}(T)$ be the division algebra of endomorphisms of Hodge structures of T. Let $' : E \to E$ be the involution given by adjunction with respect to q and let $E_0 \subset E$ be the subalgebra of E formed by q-self-adjoint endomorphisms.

Theorem 2.1.2 (Zarhin [Z])**.** *The map*

$$\epsilon : E \to \mathbb{C}, \quad e \mapsto \textit{eigenvalue of } e \textit{ on the eigenspace } T^{2,0}$$

identifies E with a subfield of $\mathbb{C}$. Moreover, E_0 is a totally real number field and the following two cases are possible:

- *$E_0 = E$ (in this case we say that T has* real multiplication*) or*
- *$E_0 \subset E$ is a purely imaginary, quadratic extension and $'$ is the restriction of complex conjugation to E (we say that T has* complex multiplication*).*

2.2. **Splitting of Hodge structures of K3 type with real multiplication.** (For this and the next section see [vG4], 2.4 and 2.5.) Let (T, h, q) be an irreducible Hodge structure of K3 type and assume that $E = \mathrm{End}_{\mathrm{Hdg}}(T)$ is a totally real number field. Note that by Theorem 2.1.2, all endomorphisms in E are q-self-adjoint.

By the theorem of the primitive element, there exists $\alpha \in E$ such that $E = \mathbb{Q}(\alpha)$. Let $d = [E : \mathbb{Q}]$. Let P be the minimal polynomial of α over $\mathbb{Q}$, denote by $\widetilde{E}$ the splitting field of P in $\mathbb{R}$. Let $G = \mathrm{Gal}(\widetilde{E}/\mathbb{Q})$ and $H = \mathrm{Gal}(\widetilde{E}/E)$. Choose $\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_d \in G$ such that

$$G = \sigma_1 H \sqcup \ldots \sqcup \sigma_d H.$$

Note that each coset $\sigma_i H$ induces a well-defined embedding $E \hookrightarrow \widetilde{E}$. In $\widetilde{E}[X]$ we get

$$P(X) = \prod_{i=1}^{d} (X - \sigma_i(\alpha))$$

and consequently

$$\begin{aligned} E \otimes_{\mathbb{Q}} \widetilde{E} &= \mathbb{Q}[X]/(P) \otimes_{\mathbb{Q}} \widetilde{E} \\ &\simeq \bigoplus_{i=1}^{d} \widetilde{E}[X]/(X - \sigma_i(\alpha)) \\ &\simeq \bigoplus_{i=1}^{d} E_{\sigma_i}. \end{aligned}$$

The symbol E_{σ_i} stands for the field $\widetilde{E}$, the index σ_i keeps track of the fact that the $\widetilde{E}$-linear extension of $E \subset \mathrm{End}_{\mathbb{Q}}(E)$ acts on E_{σ_i} via $e(x) = \sigma_i(e) \cdot x$. See Section 3.2 for another interpretation of E_{σ_i}.

In the same way, since T is a finite-dimensional E-vector space we get a decomposition

$$T_{\widetilde{E}} = T \otimes_{\mathbb{Q}} \widetilde{E} = \bigoplus_{i=1}^{d} T_{\sigma_i}.$$

This is the decomposition of $T_{\widetilde{E}}$ into eigenspaces of the $\widetilde{E}$-linear extension of the E-action on T, T_{σ_i} being the eigenspace of $e_{\widetilde{E}}$ to the eigenvalue $\sigma_i(e)$ for $e \in E$. Since each $e \in E$ is q-self-adjoint (that is $e' = e$), the decomposition is orthogonal. Let $q_{\widetilde{E}}$ be the $\widetilde{E}$-bilinear extension of q to $T_{\widetilde{E}} \times T_{\widetilde{E}}$. Using the notation

$$T_i := T_{\sigma_i} \text{ and } q_i = (q_{\widetilde{E}})_{|T_i \times T_i},$$

we have an orthogonal decomposition

$$(T_{\widetilde{E}}, q_{\widetilde{E}}) = \bigoplus_{i=1}^{d} (T_i, q_i). \tag{1}$$

2.3. **Galois action on $T_{\widetilde{E}}$.** Letting G act in the natural way on $\widetilde{E}$, we get a (only $\mathbb{Q}$-linear) Galois action on $T_{\widetilde{E}} = T \otimes_{\mathbb{Q}} \widetilde{E}$. Under this action, for $\tau \in G$ we have

$$\tau T_{\sigma_i} = T_{\tau\sigma_i}. \tag{2}$$

This is because the Galois action commutes with the $\widetilde{E}$-linear extension of any endomorphism $e \in E \subset \mathrm{End}_{\mathbb{Q}}(T)$ the latter being defined over $\mathbb{Q}$ and because for $t_i \in T_{\sigma_i}$ and $e \in E$

$$e_{\widetilde{E}}(\tau(t_i)) = \tau(e_{\widetilde{E}}(t_i)) = \tau(\sigma_i(e)t_i) = \tau(\sigma_i(e))\tau(t_i) = (\tau\sigma_i(e))\tau(t_i),$$

which means that τ permutes the eigenspaces of $e_{\widetilde{E}}$ precisely in the way we claimed. Define a homomorphism

$$\gamma : G \to \mathfrak{S}_d, \quad \tau \mapsto \{i \mapsto \tau(i) \text{ where } (\tau\sigma_i)H = \sigma_{\tau(i)}H\}. \tag{3}$$

(This describes the action of G on G/H.) With this notation, (2) reads

$$\tau T_i = T_{\tau(i)}. \tag{4}$$

Interpret T as a subspace of $T_{\widetilde{E}}$ via the natural inclusion $T \hookrightarrow T_{\widetilde{E}},\ t \mapsto t \otimes 1$. Denote by π_i the projection to T_i. For $t \in T$ and $\tau \in G$ we have $t = \tau(t)$. Write $t_i := \pi_i(t \otimes 1)$, then $t = \sum_i t_i$. Using (4) we see that

$$t_{\tau i} = \tau(t_i). \tag{5}$$

It follows that

$$\iota_i : T \to T_i, \quad t \mapsto \pi_i(t \otimes 1) \tag{6}$$

is an injective map of E-vector spaces (E acting on T_i via $\sigma_i : E \hookrightarrow \widetilde{E}$). Equation (5) can be rephrased as

$$\iota_{\tau i} = \tau \circ \iota_i. \tag{7}$$

Since q is defined over $\mathbb{Q}$, we have for $t \in T_{\widetilde{E}}$ and $\tau \in G$

$$q_{\widetilde{E}}(\tau t) = \tau q_{\widetilde{E}}(t).$$

This implies that for $t \in T$

$$q_i(\iota_i(t)) = \sigma_i q_1(\iota_1(t)). \tag{8}$$

2.4. **The special Mumford–Tate group of a Hodge structure of K3 type with real multiplication.** Zarhin [Z] also computed the special Mumford–Tate group of an irreducible Hodge structure of K3 type. Recall that for a Hodge structure $(W, h : \mathrm{U}(1) \to \mathrm{GL}(W)_{\mathbb{R}})$ the special Mumford–Tate group $\mathrm{SMT}(W)$ is the smallest linear algebraic subgroup of $\mathrm{GL}(W)$ defined over $\mathbb{Q}$ with $h(\mathrm{U}(1)) \subset \mathrm{SMT}(W)_{\mathbb{R}}$ (cf. [G]).

Assume that (T, h, q) is an irreducible Hodge structure of K3 type with real multiplication by $E = \mathrm{End}_{\mathrm{Hdg}}(T)$. We continue to use the notations of Section 2.2. Denote by Q the restriction of q_1 to $T \subset T_1$ (use the inclusion ι_1 of (6)). This is an E-valued (since H-invariant), non-degenerate, symmetric bilinear form on the E-vector space T. Denote by $\mathrm{SO}(Q)$ the E-linear algebraic group of Q-orthogonal, E-linear transformations of T with determinant 1.

Recall that for an E-variety Y the *Weil restriction* $\mathrm{Res}_{E/\mathbb{Q}}(Y)$ is the $\mathbb{Q}$-variety whose K-rational points are the $E \otimes_{\mathbb{Q}} K$-rational points of Y for any extension field $\mathbb{Q} \subset K$ (cf. [BLR]).

Theorem 2.4.1 (Zarhin, see [Z], see also [vG4], 2.8)**.** *The special Mumford–Tate group of the Hodge structure (T, h, q) with real multiplication by E is*

$$\mathrm{SMT}(T) = \mathrm{Res}_{E/\mathbb{Q}}(\mathrm{SO}(Q)).$$

Its representation on T is the natural one, where we regard T as a $\mathbb{Q}$-vector space and use that any E-linear endomorphism of T is in particular $\mathbb{Q}$-linear. After base change to $\widetilde{E}$

$$\mathrm{SMT}(T)_{\widetilde{E}} = \prod_i \mathrm{SO}((T_i), (q_i)),$$

its representation on $T_{\widetilde{E}} = \bigoplus_i (T_i)$ is the product of the standard representations.

3. Kuga–Satake varieties and real multiplication

3.1. **Kuga–Satake varieties.** Let (T, h, q) be a Hodge structure of K3 type. Kuga and Satake [KS] found a way to associate to this a polarizable $\mathbb{Q}$-Hodge structure of weight one $(V, h_s : \mathrm{U}(1) \to \mathrm{GL}(V)_{\mathbb{R}})$, in other words an isogeny class of Abelian varieties, together with an inclusion of Hodge structures

$$T \subset V \otimes V. \tag{9}$$

Set $V := C^0(q)$ where $C^0(q)$ is the even Clifford algebra of q. Define a weight one Hodge structure on V in the following way: Choose $f_1, f_2 \in (T^{2,0} \oplus T^{0,2})_{\mathbb{R}}$ such that $\mathbb{C}(f_1 + i f_2) = T^{2,0}$ and $q(f_i, f_j) = \delta_{i,j}$ (recall that $q_{|(T^{2,0} \oplus T^{0,2})_{\mathbb{R}}}$ is positive definite). Define $J : V \to V,\ v \mapsto f_1 f_2 v$, then we see that $J^2 = -\mathrm{id}$. Now we can define a homomorphism of algebraic groups

$$h_s : \mathrm{U}(1) \to \mathrm{GL}(V)_{\mathbb{R}}, \ \ \exp(xi) \mapsto \exp(xJ),$$

and this induces the Kuga–Satake Hodge structure. One can check that h_s is independent of the choice of f_1, f_2 (see [vG3, Lemma 5.5]).

It can be shown that the Kuga–Satake Hodge structure admits a polarization (cf. [vG3, Prop. 5.9]) and that there is an embedding of Hodge structures as in (9) (see [vG3, Prop. 6.3]).

3.2. **Corestriction of algebras.** Let E/K be a finite, separable extension of fields of degree d and let A be an E-algebra. We use the notations of Section 2.2, so $\widetilde{E}$ is a normal closure of E over K, $\sigma_1, \dots, \sigma_d$ is a set of representatives of G/H where $G = \mathrm{Gal}(\widetilde{E}/K)$ and $H = \mathrm{Gal}(\widetilde{E}/E)$.

For $\sigma \in G$ define the twisted $\widetilde{E}$-algebra as the ring

$$A_\sigma := A \otimes_E \widetilde{E}$$

which carries an $\widetilde{E}$-algebra structure given by

$$\lambda \cdot (a \otimes e) = a \otimes \sigma^{-1}(\lambda) e.$$

Note that $A_\sigma \simeq A \otimes_E E_\sigma$.

Let V be an E-vector space and W an $\widetilde{E}$-vector space, let $\sigma \in G$. A homomorphism of K-vector spaces $\varphi : V \to W$ is called *σ-linear* if $\varphi(\lambda v) = \sigma(\lambda)\varphi(v)$ for all $v \in V$ and $\lambda \in E$. If both, V and W are $\widetilde{E}$-vector spaces, there is a similar notion of an σ-linear homomorphism.

Lemma 3.2.1. *The map*

$$\kappa_\sigma : A \to A_\sigma, \quad a \mapsto a \otimes 1$$

is a σ-linear ring homomorphism and the pair $(A_\sigma, \kappa_\sigma)$ has the following universal property: For all $\widetilde{E}$-algebras B and for all σ-linear ring homomorphisms $\varphi : A \to B$ there exists a unique $\widetilde{E}$-algebra homomorphism $\widetilde{\varphi} : A_\sigma \to B$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\kappa_\sigma} & A_\sigma \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \widetilde{\varphi}} \\ & & B \end{array}$$

commutative.

Proof. We only check the universal property. To give a K-linear map $\alpha : A \otimes_E \widetilde{E} \to B$ is the same as to give a K-bilinear map $\beta : A \times \widetilde{E} \to B$ satisfying

$$\beta(\lambda a, e) = \beta(a, \lambda e) \tag{10}$$

for all $a \in A, e \in \widetilde{E}$ and $\lambda \in E$. These maps are related by the condition

$$\alpha(a \otimes e) = \beta(a, e).$$

Now given φ as in the lemma, we define

$$\psi : A \times \widetilde{E} \to B, \quad (a, e) \mapsto \sigma(e)\varphi(a).$$

This is a K-bilinear map which satisfies (10) and therefore, it induces a K-linear map

$$\widetilde{\varphi} : A \otimes_E \widetilde{E} \to B, \quad a \otimes e \mapsto \sigma(e)\varphi(a).$$

It is clear that $\widetilde{\varphi}$ is a ring homomorphism and that it respects the $\widetilde{E}$-algebra structures if we interpret $\widetilde{\varphi}$ as a map $\widetilde{\varphi} : A_\sigma \to B$. The uniqueness of this map is immediate. □

Remark. (i) The lemma shows that up to unique $\widetilde{E}$-algebra isomorphism, the twisted algebra A_{σ_i} depends only on the coset $\sigma_i H$. Indeed, for $\sigma \in \sigma_i H$ the inclusion $A \hookrightarrow A_{\sigma_i}$ is σ-linear because σ and σ_i induce the same embedding of E into $\widetilde{E}$. By the lemma, there exists an $\widetilde{E}$-algebra isomorphism $\alpha_{\sigma,\sigma_i} : A_\sigma \xrightarrow{\sim} A_{\sigma_i},\ a \otimes e \mapsto \sigma(e) \cdot (a \otimes 1) = a \otimes \sigma_i^{-1}\sigma(e)$.

(ii) In Section 2.2 we were in the situation $E = \mathbb{Q}(\alpha)$. There we discussed the splitting $E \otimes_{\mathbb{Q}} \widetilde{E} \simeq \bigoplus_i \widetilde{E}[X]/(X - \sigma_i(\alpha)) \simeq \bigoplus_i E_{\sigma_i}$ and we used the symbol E_{σ_i} for the field $\widetilde{E}$ with E-action via $e(x) = \sigma_i(e) \cdot x$. This is precisely our twisted $\widetilde{E}$-algebra E_{σ_i} on which E acts via the inclusion κ_{σ_i}.

For $\tau \in G$ there is a unique τ-linear ring isomorphism $\tau : A_{\sigma_i} \to A_{\sigma_{\tau i}}$ which extends the identity on $A \subset A_{\sigma_i}$ (in the sense that $\tau \circ \kappa_{\sigma_i} = \kappa_{\sigma_{\tau i}}$). This map is given as the composition of the following two maps: First apply the identity map $A_{\sigma_i} \to A_{\tau\sigma_i},\ a \otimes e \mapsto a \otimes e$ which is a τ-linear ring isomorphism. Then apply the isomorphism $\alpha_{\tau\sigma_i,\sigma_{\tau i}}$ (by definition of the G-action on $\{1, \dots, d\}$ we have $\tau\sigma_i \in \sigma_{\tau i}H$). On simple tensors the map τ takes the form

$$a \otimes e \mapsto a \otimes \sigma_{\tau i}^{-1}\tau\sigma_i(e). \tag{11}$$

These maps induce a natural action of G on

$$Z_G(A) := A_{\sigma_1} \otimes_{\widetilde{E}} \dots \otimes_{\widetilde{E}} A_{\sigma_d}$$

where

$$\begin{aligned}(12)\quad & \tau((a_1 \otimes e_1) \otimes \dots \otimes (a_d \otimes e_d)) \\ &= \left(a_{\tau^{-1}1} \otimes \sigma_1^{-1}\tau\sigma_{\tau^{-1}1}(e_{\tau^{-1}1})\right) \otimes \dots \otimes \left(a_{\tau^{-1}d} \otimes \sigma_d^{-1}\tau\sigma_{\tau^{-1}d}(e_{\tau^{-1}d})\right).\end{aligned}$$

Definition 3.2.2 ([D], §8, Def. 2 or [T], 2.2)**.** The *corestriction of A to K* is the K-algebra of G-invariants in $Z_G(A)$

$$\mathrm{Cores}_{E/K}(A) := Z_G(A)^G.$$

Remark. (i) By [D, §8, Cor. 1] there is a natural isomorphism

$$\mathrm{Cores}_{E/K}(A) \otimes_K \widetilde{E} \simeq Z_G(A)$$

In particular, with $d = [E : K]$ one gets $\dim_K \mathrm{Cores}_{E/K}(A) = (\dim_E(A))^d$.

(ii) Let $X = \mathrm{Spec}(A)$ for a commutative L-algebra A. Then for any K-algebra B we get a chain of isomorphisms, functorial in B

$$\begin{aligned}\mathrm{Hom}_{K-\mathrm{Alg}}(\mathrm{Cores}_{E/K}(A), B) &\simeq \left(\mathrm{Hom}_{\widetilde{E}-\mathrm{Alg}}(Z_G(A), B \otimes_K \widetilde{E})\right)^G \\ &\simeq \mathrm{Hom}_{E-\mathrm{Alg}}(A, B \otimes_K E).\end{aligned}$$

Here, the last isomorphism is given by composing $f \in \left(\mathrm{Hom}_{\widetilde{E}-\mathrm{Alg}}(Z_G(A), B \otimes_K \widetilde{E})\right)^G$ with the inclusion $j : A \hookrightarrow Z_G(A),\ a \mapsto \kappa_{\sigma_1}(a) \otimes 1 \otimes \dots \otimes 1$. (The image of this composition is contained in the H-invariant part of $B \otimes_K \widetilde{E}$ which is $B \otimes_K E$.) This map is an isomorphism, since $Z_G(A)$ is generated as an $\widetilde{E}$-algebra by elements of the form $\sigma \circ j(a)$ with $a \in A$ and $\sigma \in G$.

It follows that

$$\mathrm{Res}_{E/K}(\mathrm{Spec}(A)) \simeq \mathrm{Spec}(\mathrm{Cores}_{E/K}(A)),$$

i.e. the Weil restriction of affine E-schemes is the same as the corestriction of commutative E-algebras.

3.3. **The decomposition theorem.** We will assume from now to the end of the section that (T, h, q) is an irreducible Hodge structure of K3 type with $E = \mathrm{End}_{\mathrm{Hdg}}(T)$ a totally real number field.

Recall that in this case T is an E-vector space which carries a natural E-valued quadratic form Q (see 2.4). Let $C^0(Q)$ be the even Clifford algebra of Q over E. It was van Geemen (see [vG4, Prop. 6.3]) who discovered that the algebra $\mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))$ appears as a sub-Hodge structure in the Kuga–Satake Hodge structure of (T, h, q). We are going to show that this contains all information on the Kuga–Satake Hodge structure.

Theorem 3.3.1. *Denote by (V, h_s) the Kuga–Satake Hodge structure of (T, h, q).*

(i) *The special Mumford–Tate group of (V, h_s) is the image of $\mathrm{Res}_{E/\mathbb{Q}}(\mathrm{Spin}(Q))$ in $\mathrm{Spin}(q)$ under a morphism m of rational algebraic groups which after base change to $\widetilde{E}$ becomes*

$$m_{\widetilde{E}} : \mathrm{Spin}(q_1) \times \ldots \times \mathrm{Spin}(q_d) \to \mathrm{Spin}(q)_{\widetilde{E}}, \quad (v_1, \ldots, v_d) \mapsto v_1 \cdot \ldots \cdot v_d.$$

(ii) *Let $W := \mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))$. Then W can be canonically embedded in V and the image is $\mathrm{SMT}(V)$-stable and therefore, it is a sub-Hodge structure. Furthermore, there is a (non-canonical) isomorphism of Hodge structures*

$$V \simeq W^{2^{d-1}}.$$

(iii) *We have*

$$\mathrm{End}_{\mathrm{Hdg}}(W) = \mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))$$

and consequently

$$\mathrm{End}_{\mathrm{Hdg}}(V) = \mathrm{Mat}_{2^{d-1}}\big(\mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))\big).$$

The proof will be given in Section 3.5. The theorem tells us that the Kuga–Satake variety A of (T, h, q) is isogenous to a self-product $B^{2^{d-1}}$ of an Abelian variety B with $\mathrm{End}_{\mathbb{Q}}(B) = \mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))$ and therefore, it proves Theorem 1.

Note that B is not simple in general. We will see examples below where B decomposes further.

3.4. **Galois action on $C(q)_{\widetilde{E}}$.** By Section 2.2 there is a decomposition

$$(T, q)_{\widetilde{E}} = \bigoplus_{i=1}^{d} (T_i, q_i).$$

This in turn yields an isomorphism

$$C(q)_{\widetilde{E}} \simeq C(q_{\widetilde{E}}) \simeq C(q_1) \widehat{\otimes}_{\widetilde{E}} \ldots \widehat{\otimes}_{\widetilde{E}} C(q_d).$$

Here, the symbol $\widehat{\otimes}$ denotes the graded tensor product of algebras, which on the level of vector spaces is just the usual tensor product, but which twists the algebra structure by a suitable sign (see [vG3, ??]).

Decompose $C(q_i) = C^0(q_i) \oplus C^1(q_i)$ in the even and the odd part. If we forget the algebra structure and only look at $\widetilde{E}$-vector spaces, we get

$$C(q)_{\widetilde{E}} = \bigoplus_{\mathbf{a}\in\{0,1\}^d} C^{a_1}(q_1) \otimes_{\widetilde{E}} \ldots \otimes_{\widetilde{E}} C^{a_d}(q_d).$$

For $\mathbf{a} = (a_1, \ldots, a_d) \in \{0,1\}^d$ define

$$C^{\mathbf{a}}(q) = C^{a_1}(q_1) \otimes \ldots \otimes C^{a_d}(q_d).$$

We introduced an action of $G = \mathrm{Gal}(\widetilde{E}/\mathbb{Q})$ on $\{1, \ldots, d\}$ (see (3)). This induces an action

$$G \times \{0,1\}^d \to \{0,1\}^d, \ \ (\tau, (a_1, \ldots, a_d)) \mapsto (a_{\tau^{-1}1}, \ldots, a_{\tau^{-1}d}). \tag{13}$$

The next lemma describes the Galois action on $C(q)_{\widetilde{E}}$.

Lemma 3.4.1. (i) *Via the map*

$$C(q_i) \subset C(q)_{\widetilde{E}}, \ v_i \mapsto 1 \otimes \ldots \otimes v_i \otimes \ldots \otimes 1$$

we interpret $C(q_i)$ as a subalgebra of $C(q)_{\widetilde{E}}$. Then the restriction of $\tau \in G$ to $C(q_i)$ induces an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Q}$-algebras

$$\tau : (C(q_i)) \xrightarrow{\sim} C(q_{\tau i}).$$

(ii) *For $\tau \in G$ and $\mathbf{a} \in \{0,1\}^d$ we get*

$$\tau(C^{\mathbf{a}}(q)) = C^{\tau\mathbf{a}}(q).$$

Proof. Tensor the natural inclusion $T \hookrightarrow C(q)$ with $\widetilde{E}$ to get a G-equivariant inclusion

$$T \otimes_{\mathbb{Q}} \widetilde{E} = \bigoplus_{i=1}^{d} T_i \to C(q)_{\widetilde{E}}.$$

Using (4), we find for $t_i \in C(q_i)$ that $\tau(t_i) \in C(q_{\tau(i)})$. Now, $C(q_i)$ is spanned as a $\mathbb{Q}$-algebra by products of the form

$$t_1 \cdot \ldots \cdot t_k = \pm(1 \otimes \ldots \otimes t_1 \otimes \ldots \otimes 1) \cdot \ldots \cdot (1 \otimes \ldots \otimes t_k \otimes \ldots \otimes 1)$$

for $t_1, \ldots, t_k \in T_i$. Since G acts by $\mathbb{Q}$-algebra homomorphisms on $C(q)_{\widetilde{E}}$, this implies (i).

Item (ii) is an immediate consequence of (i): The space $C^{\mathbf{a}}(q)$ is spanned as $\mathbb{Q}$-vector space by products of the form $v_1 \cdot \ldots \cdot v_d = \pm v_1 \otimes \ldots \otimes v_d$ with $v_i \in C^{a_i}(q_i)$. Then use again, that G acts by $\mathbb{Q}$-algebra homomorphisms. □

Lemma 3.4.2. *For $i \in \{1, \ldots, d\}$ the twisted algebra $C^0(Q)_{\sigma_i}$ is canonically isomorphic as an $\widetilde{E}$-algebra to $C^0(q_i)$. Thus*

$$Z_G(C^0(Q)) \simeq C^0(q_1) \otimes_{\widetilde{E}} \ldots \otimes_{\widetilde{E}} C^0(q_d).$$

On both sides there are natural G-actions: On the left hand side G acts via the action introduced in (12), whereas on the right hand side it acts via the restriction of its action on $C(q)_{\widetilde{E}}$ (use Lemma 3.4.1). Then the above isomorphism is G-equivariant.

Proof. Fix $i \in \{1, \ldots, d\}$. The composition of the canonical inclusion $C^0(Q) \subset C^0(q_1) \simeq C^0(Q)_{\widetilde{E}}$ with the restriction to $C^0(Q)$ of the map $\sigma_i : C(q_1) \to C(q_i)$ from Lemma 3.4.1 induces a σ_i-linear ring homomorphism

$$\varphi_i : C^0(Q) \hookrightarrow C^0(q_i).$$

By Lemma 3.2.1 we get an $\widetilde{E}$-algebra homomorphism

$$\widetilde{\varphi}_i : C^0(Q)_{\sigma_i} \to C^0(q_i).$$

Recall that there are inclusions $\iota_i : T \hookrightarrow T_i$ (see (6)) which satisfy $\tau \circ \iota_i = \iota_{\tau i}$ (see (7)). Let $t_1, \ldots, t_m \in T$ such that $\iota_1(t_1), \ldots, \iota_1(t_m)$ form a q_1-orthogonal basis of T_1. Then the vectors $\iota_i(t_1), \ldots, \iota_i(t_m)$ form a q_i-orthogonal basis of T_i (use (8)). By definition of $\widetilde{\varphi}_i$

$$\widetilde{\varphi}_i \left(\iota_1(t_1)^{i_1} \cdot \ldots \cdot \iota_1(t_m)^{i_d}\right) = \iota_i(t_1)^{i_1} \cdot \ldots \cdot \iota_i(t_m)^{i_m}. \tag{14}$$

This implies that $\widetilde{\varphi}_i$ maps an $\widetilde{E}$-basis of $C^0(Q)_{\sigma_i}$ onto an $\widetilde{E}$-basis of $C^0(q_i)$, whence $\widetilde{\varphi}_i$ is an isomorphism of $\widetilde{E}$-algebras.

As for the G-equivariance, we have to check that for all $\tau \in G$ the diagram

$$\begin{array}{ccc} C^0(Q)_{\sigma_i} & \xrightarrow{\widetilde{\varphi}_i} & C^0(q_i) \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau} \\ C^0(Q)_{\sigma_{\tau i}} & \xrightarrow{\widetilde{\varphi}_{\tau i}} & C^0(q_{\tau i}) \end{array}$$

is commutative. It is enough to check this on an $\widetilde{E}$-basis of $C^0(Q)_{\sigma_i}$ because the vertical maps are both τ-linear whereas the horizontal ones are $\widetilde{E}$-linear. Since $\tau : C^0(Q)_{\sigma_i} \to C^0(Q)_{\sigma_{\tau i}}$ was defined as the extension of the identity map on $C^0(Q) \subset C^0(Q)_{\sigma_i}$, we have

$$\begin{aligned} \widetilde{\varphi}_{\tau i} \circ \tau \left(\iota_1(t_1)^{i_1} \cdot \ldots \cdot \iota_1(t_m)^{i_m}\right) &= \widetilde{\varphi}_{\tau i} \left(\iota_1(t_1)^{i_1} \cdot \ldots \cdot \iota_1(t_m)^{i_m}\right) \\ &= \iota_{\tau i}(t_1)^{i_1} \cdot \ldots \cdot \iota_{\tau i}(t_m)^{i_m} \\ &= (\tau \circ \iota_i)(t_1)^{i_1} \cdot \ldots \cdot (\tau \circ \iota_i)(t_m)^{i_m} \\ &= \tau \left(\iota_i(t_1)^{i_1} \cdot \ldots \cdot \iota_i(t_m)^{i_m}\right) \\ &= \tau \circ \widetilde{\varphi}_i \left(\iota_1(t_1)^{i_1} \cdot \ldots \cdot \iota_m(t_m)^{i_m}\right). \end{aligned}$$

This completes the proof of the lemma. □

3.5. **Proof of the decomposition theorem.** Let K be a field and (U, r) be a quadratic K-vector space. Recall that the spin group of r comes with two natural representations:

First there is the covering representation $\rho : \mathrm{Spin}(r) \to \mathrm{SO}(r)$ which over an extension field $K \subset L$ maps $y \in \mathrm{Spin}(r)(L) = \{x \in (C^0(r) \otimes_K L)^* \mid x\iota(x) = 1 \text{ and } xUx^{-1} \subset U\}$ to the endomorphism $U \to U,\ u \mapsto xux^{-1}$. Here, $\iota : C(r) \to C(r)$ is the natural involution of the Clifford algebra.

Secondly, the spin representation realizes $\mathrm{Spin}(r)$ as a subgroup of $\mathrm{GL}(C^0(r))$ by sending $y \in \mathrm{Spin}(r)(L)$ to the endomorphism of $C^0(r)$ given by $x \mapsto y \cdot x$.

Proof of (i). By [vG3, Prop. 6.3], there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{U}(1) & \xrightarrow{h_s} & \mathrm{Spin}(q)_{\mathbb{R}} & \longrightarrow & \mathrm{GL}(C^0(q))_{\mathbb{R}} \\ \| & & \downarrow \rho & & \\ \mathrm{U}(1) & \xrightarrow[h]{} & \mathrm{SO}(q)_{\mathbb{R}} & \longrightarrow & \mathrm{GL}(T)_{\mathbb{R}}. \end{array} \tag{15}$$

(Van Geemen works with the Mumford–Tate group, therefore he gets a factor t^2 in 6.3.2. This factor is 1 if one restricts the attention to the special Mumford–Tate group, moreover it is then clear that $h_s(\mathbb{C}^*) \subset \mathrm{CSpin(q)} = \{v \in C^0(q)^* \mid vTv^{-1} \subset T\}$ implies $h_s(\mathrm{U}(1)) \subset \mathrm{Spin}(q)$.)

Claim: There is a Cartesian diagram

$$\begin{array}{ccc} \mathrm{SMT}(V) & \longrightarrow & \mathrm{Spin}(q) \\ {\scriptstyle \rho_{|\,\mathrm{SMT}(V)}} \downarrow & & \downarrow \rho \\ \mathrm{SMT}(T) & \longrightarrow & \mathrm{SO}(q). \end{array}$$

where the horizontal maps are appropriate factorizations of the inclusions $\mathrm{SMT} \subset \mathrm{GL}$ whose existence is guaranteed by (15).

Proof of the claim. It is clear by looking at (15) and at the definition of the special Mumford–Tate group that

$$\mathrm{SMT}(V) \subset \mathrm{SMT}(T) \times_{\mathrm{SO}(q)} \mathrm{Spin}(q).$$

In the same way we see that

$$\mathrm{SMT}(T) \subset \rho(\mathrm{SMT}(V))$$

and hence we have a chain of inclusions

$$\mathrm{SMT}(V) \subset \mathrm{SMT}(T) \times_{\mathrm{SO}(q)} \mathrm{Spin}(q) \subset \rho(\mathrm{SMT}(V)) \times_{\mathrm{SO}(q)} \mathrm{Spin}(q).$$

But over any field, the kernel of ρ consists of $\{\pm 1\} \subset \mathrm{SMT}(V)$ (because $h_s(-1) = -1$) and thus

$$\mathrm{SMT}(V) = \rho(\mathrm{SMT}(V)) \times_{\mathrm{SO}(q)} \mathrm{Spin}(q).$$

This proves the claim. (Claim)$\square$

To continue the proof of (i) we have to define the morphism of rational algebraic groups

$$m : \mathrm{Res}_{E/\mathbb{Q}}(\mathrm{Spin}(Q)) \to \mathrm{Spin}(q).$$

For that sake, note first that there is a natural isomorphism of $\widetilde{E}$-algebras

$$(16)\qquad \begin{aligned} C^0(Q) \otimes_{\mathbb{Q}} \widetilde{E} &\simeq C^0(Q) \otimes_E (E \otimes_{\mathbb{Q}} \widetilde{E}) \\ &\simeq \bigoplus_i C^0(Q) \otimes_E E_{\sigma_i} \\ &\simeq \bigoplus_i C^0(Q)_{\sigma_i} \\ &\simeq C^0(q_1) \oplus \ldots \oplus C^0(q_d) \end{aligned}$$

where we use the notations of Section 3.2 and for the last identification Lemma 3.4.2. Consider the natural G-action on $C^0(q_1) \oplus \ldots \oplus C^0(q_d)$ given by

$$(\tau, (v_1, \ldots, v_d)) \mapsto (\tau v_{\tau^{-1}1}, \ldots, \tau v_{\tau^{-1}d}).$$

On $C^0(Q) \otimes_{\mathbb{Q}} \widetilde{E}$, the Galois group G acts by its natural action on $\widetilde{E}$. Then the identification made in (16) is G-equivariant and we get an isomorphism of $\mathbb{Q}$-vector spaces

$$C^0(Q) \simeq \left(C^0(q_1) \oplus \ldots \oplus C^0(q_d)\right)^G, \quad v \mapsto (\sigma_1(v), \ldots, \sigma_d(v)).$$

Now, look at the morphism of $\widetilde{E}$-affine spaces

$$C^0(q_1) \oplus \ldots \oplus C^0(q_d) \to C^0(q)_{\widetilde{E}}, \quad (v_1, \ldots, v_d) \mapsto v_1 \cdot \ldots \cdot v_d.$$

This morphism is G-equivariant on the $\widetilde{E}$-points and hence it comes from a morphism of $\mathbb{Q}$-varieties

$$\operatorname{Res}_{E/\mathbb{Q}} C^0(Q) \to C^0(q).$$

The restriction of this latter to $\operatorname{Res}_{E/\mathbb{Q}}(\operatorname{Spin}(Q))$ is the morphism m we are looking for. It is a morphism of algebraic groups which after base change to $\widetilde{E}$ takes the form

$$m_{\widetilde{E}} : \operatorname{Res}_{E/\mathbb{Q}}(\operatorname{Spin}(Q))_{\widetilde{E}} \simeq \operatorname{Spin}(q_1) \times \ldots \times \operatorname{Spin}(q_d) \to \operatorname{Spin}(q)_{\widetilde{E}}, \quad (v_1, \ldots, v_d) \mapsto v_1 \cdot \ldots \cdot v_d.$$

It remains to show that the image of m in $\operatorname{Spin}(q)$ is $\operatorname{SMT}(V)$. Using the claim we have to show that the following diagram exists and that it is Cartesian

$$(17)\qquad \begin{array}{ccc} \operatorname{im}(m) & \longrightarrow & \operatorname{Spin}(q) \\ {\scriptstyle \rho_{|\operatorname{im}(m)}}\downarrow & & \downarrow{\scriptstyle \rho} \\ \operatorname{Res}_{E/\mathbb{Q}}(\operatorname{SO}(Q)) & \longrightarrow & \operatorname{SO}(q). \end{array}$$

Here, the lower horizontal map is the one coming from Zarhin's Theorem 2.4.1.

It is enough to study (17) on $\overline{\mathbb{Q}}$-points. It is easily seen that over $\widetilde{E} \subset \overline{\mathbb{Q}}$ the composition $\rho \circ m$ factorizes over

$$\rho_1 \times \ldots \times \rho_d : \operatorname{Spin}(q_1) \times \ldots \times \operatorname{Spin}(q_d) \to \operatorname{SO}(q_1) \times \ldots \times \operatorname{SO}(q_d) \simeq \operatorname{Res}_{E/\mathbb{Q}}(\operatorname{SO}(Q))_{\widetilde{E}} \subset \operatorname{SO}(q)_{\widetilde{E}}.$$

This shows that (17) exists. Moreover we see that $\rho_{|\operatorname{im}(m)}$ surjects onto $\operatorname{SMT}(T)(\overline{\mathbb{Q}})$ because $\rho_1 \times \ldots \times \rho_d$ does so. Since $\ker(\rho) = \{\pm 1\} \subset \operatorname{im}(m)$, the diagram (17) is Cartesian. This completes the proof of (i). (i)□

Proof of (ii). Choose $\mathbf{a}_0 = (0,\dots,0),\dots,\mathbf{a}_r \in \{0,1\}^d$ such that

$$\left\{\mathbf{a} \in \{0,1\}^d \mid \sum_i a_i \equiv 0\ (2)\right\} = G\mathbf{a}_0 \sqcup \ldots \sqcup G\mathbf{a}_r,$$

where G acts on $\{0,1\}^d$ via the action introduced in (13). Let $G_{\mathbf{a}_j} \subset G$ be the stabilizer of $\mathbf{a}_j$. Then

$$\begin{aligned} C^0(q)_{\widetilde{E}} &= \bigoplus_{j=0}^{r}\left(\bigoplus_{[\tau]\in G/G_{\mathbf{a}_j}} C^{\tau\mathbf{a}_j}(q)\right) \\ &= \bigoplus_{j=0}^{r} D^{\mathbf{a}_j} \end{aligned} \tag{18}$$

with $D^{\mathbf{a}_j} = \bigoplus_{[\tau]\in G/G_{\mathbf{a}_j}} C^{\tau\mathbf{a}_j}(q)$.

By Lemma 3.4.1 this is a decomposition of G-modules. Moreover, recall that $\mathrm{Spin}(q_1)\times\ldots\times\mathrm{Spin}(q_d)$ acts on $C^0(q)_{\widetilde{E}}$ by sending $(v_1,\dots,v_d)$ to the endomorphism of $C^0(q)_{\widetilde{E}}$ given by left multiplication with $m(v_1,\dots,v_d) = v_1\cdot\ldots\cdot v_d$. Under this action each $C^{\mathbf{a}}(q)$ is $(\mathrm{Spin}(q_1)\times\ldots\times\mathrm{Spin}(q_d))$-stable. Thus, by (i) the decomposition (18) is also a decomposition of $\mathrm{SMT}(V)(\widetilde{E})$-modules. Hence, by passing to G-invariants, (18) leads to a decomposition of Hodge structures.

Denote by

$$R := D^{\mathbf{a}_0} = C^{\mathbf{a}_0}(q) = C^0(q_1)\otimes_{\widetilde{E}}\ldots\otimes_{\widetilde{E}} C^0(q_d).$$

By Lemma 3.4.2, using the notations of Section 3.2, we have

$$R = Z_G(C^0(Q))$$

as G-modules and hence $R^G = \mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))$. Thus we have recovered

$$W = \mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q)) \subset C^0(q) = V$$

as a sub-Hodge structure. We now prove that after passing to G-invariants, the remaining summands in (18) are isomorphic to sums of copies of W.

Denote by $d_j = \sharp(G/G_{\mathbf{a}_j})$ and choose a set of representatives $\mu_1,\dots,\mu_{d_j}$ of $G/G_{\mathbf{a}_j}$ in G. We consider three group actions on $R^{\oplus d_j}$:

- First there is a natural $(\mathrm{Spin}(q_1)\times\ldots\times\mathrm{Spin}(q_d))$-action which is just the diagonal action of the one on R.
- Let $\alpha : G\times R^{\oplus d_j} \to R^{\oplus d_j}$ be the diagonal action of the G-action on R.
- Finally define the G-action β by

$$\beta : \begin{cases} G\times\bigoplus_{l=1}^{d_j} R_{[\mu_l]} \to \bigoplus_{l=1}^{d_j} R_{[\mu_l]} \\ (\tau,(r_{[\mu_1]},\dots,r_{[\mu_d]})) \mapsto (\tau r_{[\tau^{-1}\mu_1]},\dots,\tau r_{[\tau^{-1}\mu_{d_j}]}). \end{cases}$$

Now we will proceed in two steps:

(a) We show that $D^{\mathbf{a}_j}$ is isomorphic as G-module and as $(\mathrm{Spin}(q_1) \times \ldots \times \mathrm{Spin}(q_d))$-module to $R^{\oplus d_j}$ where G acts on the latter via β.

(b) We show that $R^{\oplus d_j}$ is isomorphic as G-module and as $(\mathrm{Spin}(q_1) \times \ldots \times \mathrm{Spin}(q_d))$-module with G acting via α to $R^{\oplus d_j}$ with G acting via β.

Note that neither of these two isomorphisms is canonical. Once (a) and (b) are proved, we have an isomorphism

$$V_{\widetilde{E}} = C^0(q)_{\widetilde{E}} \simeq R^{\oplus 2^{d-1}}$$

of G-modules and of $\mathrm{SMT}(V)(\widetilde{E})$-modules, G acting diagonally on the right hand side. Here we use that

$$\sum_j d_j = \sharp \left\{ \mathbf{a} \in \{0,1\}^d \mid \sum_i a_i \equiv 0 \ (2) \right\} = 2^{d-1}.$$

The proof of (ii) is then accomplished by passing to G-invariants.

Proof of (a). Denote by F_j the field $\widetilde{E}^{G_{\mathbf{a}_\mathbf{j}}}$. As $C^{\mathbf{a}_j}(q) \subset D^{\mathbf{a}_j}$ is $G_{\mathbf{a}_j}$-stable, $C^{\mathbf{a}_j}(q) = W_j \otimes_{F_j} \widetilde{E}$ for some F_j-vector space W_j. Since $C^{\mathbf{a}_j}$ contains units in $C(q)_{\widetilde{E}}$, so does $W_j \subset C^{\mathbf{a}_j}$. (Very formally: There is a linear map $C^{\mathbf{a}_j} \to \mathrm{End}(C(q)_{\widetilde{E}}),\ w \mapsto \{v \mapsto v \cdot w\}$ which is defined over F_j. The image of this map over $\widetilde{E}$ intersects the Zariski-open subset of automorphisms of $C(q)_{\widetilde{E}}$, hence this must happen already over F_j.)

Choose a unit $w_j \in W_j$. Then for $\tau \in G$, since w_j is $G_{\mathbf{a}_j}$-invariant, $\tau w_j \in C^{\tau \mathbf{a}_j}(q)$ depends only on the coset $\tau G_{\mathbf{a}_j}$ and is again a unit in $C(q)_{\widetilde{E}}$.

Define an isomorphism of $\widetilde{E}$-vector spaces

$$\varphi : \begin{cases} D^{\mathbf{a}_j} = \bigoplus_{l=1}^{d_j} C^{\mu_l \mathbf{a}_j}(q) \to \bigoplus_{l=1}^{d_j} R_{[\mu_l]} \\ \qquad (v_{\mu_1}, \ldots, v_{\mu_{d_j}}) \mapsto (v_{\mu_1} \cdot \mu_1(w_j), \ldots, v_{\mu_{d_j}} \cdot \mu_{d_j}(w_j)). \end{cases}$$

This map is clearly $(\mathrm{Spin}(q_1) \times \ldots \times \mathrm{Spin}(q_d))$-equivariant since this group acts by multiplication on the left whereas we multiply on the right.

As for the G-equivariance (G acting via β on the right hand side), we find for $(v_{[\mu_1]}, \ldots, v_{[\mu_{d_j}]}) \in D^{\mathbf{a}_j}$ and $\tau \in G$:

$$\begin{aligned} \varphi\big(\tau(v_{[\mu_1]}, \ldots, v_{[\mu_{d_j}]})\big) &= \varphi\big(\tau v_{[\tau^{-1}\mu_1]}, \ldots, \tau v_{[\tau^{-1}\mu_d]}\big) \\ &= \big(\tau v_{[\tau^{-1}\mu_1]} \cdot \mu_1 w_j, \ldots, \tau v_{[\tau^{-1}\mu_{d_j}]} \cdot \mu_{d_j} w_j\big) \\ &= \big(\tau(v_{[\tau^{-1}\mu_1]} \cdot \tau^{-1}\mu_1 w_j), \ldots, \tau(v_{[\tau^{-1}\mu_{d_j}]} \cdot \tau^{-1}\mu_{d_j} w_j)\big) \\ &= \beta\big(\tau, (v_{\mu_1} \cdot \mu_1 w_j, \ldots, v_{\mu_d} \cdot \mu_{d_j} w_j)\big) \\ &= \beta\big(\tau, \varphi(v_{[\mu_1]}, \ldots, v_{[\mu_{d_j}]})\big). \end{aligned}$$

Here we used in the penultimate equality that σw_j depends only on the coset $\sigma G_{\mathbf{a}_j}$. This proves (a). (a)$\square$

Proof of (b). Choose a $\mathbb{Q}$-basis $f_1, \ldots, f_{d_j}$ of F_j. For $i = 1, \ldots, d_j$ define an $\widetilde{E}$-vector space homomorphism by

$$\psi_i : \begin{cases} R \hookrightarrow \bigoplus_{l=1}^{d_j} R_{[\mu_l]} \\ r \mapsto \big(\mu_1(f_i) \cdot r, \ldots, \mu_{d_j}(f_i) \cdot r\big). \end{cases}$$

As $(\mathrm{Spin}(q_1) \times \ldots \times \mathrm{Spin}(q_d))(\widetilde{E})$ acts by $\widetilde{E}$-linear automorphisms on R, the ψ_i are equivariant for the Spin-action.

Let's show that ψ_i is G-equivariant, G acting on the right hand side via β. For $\tau \in G$ and $r \in R$ we get

$$\begin{aligned} \psi_i(\tau r) &= \big(\mu_1(f_i) \cdot \tau r, \ldots, \mu_{d_j}(f_i) \cdot \tau r\big) \\ &= \big(\tau(\tau^{-1}\mu_1(f_i) \cdot r), \ldots, \tau(\tau^{-1}\mu_{d_j}(f_i) \cdot r)\big) \\ &= \beta\big(\tau, (\mu_1(f_i) \cdot r, \ldots, \mu_{d_j}(f_i) \cdot r)\big) \\ &= \beta(\tau, \psi_i(r)). \end{aligned}$$

Once more, we used the fact that σf_i depends only on the coset $\sigma G_{\mathbf{a}_j}$.

Finally, using Artin's independence of characters (see [La, Thm. VI.4.1]), we get

$$\det((\mu_l(f_i))_{l,i}) \neq 0.$$

Consequently, the map

$$\oplus_{i=1}^{d_j} \psi_i : R^{\oplus d_j} \to R^{\oplus d_j}$$

is an isomorphism which has the equivariance properties we want and (b) is proved. (ii)$\square$

Proof of (iii). Using that endomorphisms of Hodge structures are precisely those endomorphisms which commute with the special Mumford–Tate group, we have to show that

$$\mathrm{End}_{\mathrm{SMT}(V)}(W) = \mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q)).$$

Denote by $\mathfrak{g}$ the Lie algebra of $\mathrm{SMT}(V)$. Then

$$\begin{aligned} \mathrm{End}_{\mathrm{SMT}(V)}(W) &= \mathrm{End}_{\mathfrak{g}}(W) \\ &= \{f \in \mathrm{End}_{\mathbb{Q}}(W) \mid Xf - fX = 0 \text{ for all } X \in \mathfrak{g}\}. \end{aligned}$$

Since for any field extension $K/\mathbb{Q}$ we have $\mathrm{Lie}(\mathrm{SMT}(V)_K) = \mathfrak{g} \otimes_{\mathbb{Q}} K$ this implies that

$$\mathrm{End}_{\mathrm{SMT}(V)_K}(W_K) = \mathrm{End}_{\mathrm{SMT}(V)}(W) \otimes_{\mathbb{Q}} K. \tag{19}$$

Now $\mathrm{SMT}(V)(\widetilde{E}) = \mathrm{Spin}(q_1) \times \ldots \times \mathrm{Spin}(q_d)(\widetilde{E})$ acts on $W_{\widetilde{E}} = C^0(q_1) \otimes \ldots \otimes C^0(q_d)$ by factorwise left multiplication:

$$\big((v_1, \ldots, v_d), w_1 \otimes \ldots \otimes w_d\big) \mapsto (v_1 \cdot w_1) \otimes \ldots \otimes (v_d \cdot w_d).$$

Therefore, using multiplication on the right, we get an inclusion

$$\big(C^0(q_1)\otimes\ldots\otimes C^0(q_d)\big)^{\mathrm{op}} \hookrightarrow \mathrm{End}_{\mathrm{SMT}(V)(\widetilde{E})}(W_{\widetilde{E}}), \quad w\mapsto \{w'\mapsto w'\cdot w\}.$$

Now, $(C^0(q_1)\otimes\ldots\otimes C^0(q_d))^{\mathrm{op}} \simeq C^0(q_1)^{\mathrm{op}}\otimes\ldots\otimes C^0(q_d)^{\mathrm{op}} \simeq C^0(q_1)\otimes\ldots\otimes C^0(q_d)$ and hence passing to G-invariants we have an inclusion

$$\mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q)) \hookrightarrow \mathrm{End}_{\mathrm{SMT}(V)(\mathbb{Q})}(W). \tag{20}$$

We will now show that this is an isomorphism over $\widetilde{E}$. Using (19) and comparing dimensions this will prove (iii).

To show that (20) is an isomorphism over $\widetilde{E}$ we have to determine the $\mathrm{Spin}(q_1)\times\ldots\times\mathrm{Spin}(q_d)$-invariants in

$$\mathrm{End}_{\widetilde{E}}\big(C^0(q_1)\otimes\ldots\otimes C^0(q_d)\big) = \mathrm{End}_{\widetilde{E}}\, C^0(q_1)\otimes\ldots\otimes \mathrm{End}_{\widetilde{E}}\, C^0(q_d).$$

Using the next lemma inductively, this is equal to

$$\mathrm{End}_{\mathrm{Spin}(q_1)}\, C^0(q_1)\otimes\ldots\otimes \mathrm{End}_{\mathrm{Spin}(q_d)}\, C^0(q_d).$$

Now by [vG3, Lemma 6.5], $\mathrm{End}_{\mathrm{Spin}(q_i)}\, C^0(q_i) = C^0(q_i)$. This proves (iii). □

Lemma 3.5.1. *Let G and H be two reductive linear algebraic groups over a field K of characteristic 0. Let M resp. N be finite-dimensional representations over K of G resp. H. Then*

$$(M\otimes_K N)^{G\times H} = M^G\otimes_K N^H.$$

Proof. Decompose $M=\bigoplus_i M_i$ and $N=\bigoplus_j N_j$ in irreducible representations. Then $M_i\otimes N_j$ is an irreducible representation of $G\times H$ since fixing $0\neq m_0\in M_i$ and $0\neq n_0\in N_i$ the orbit $(G\times H)m_0\otimes n_0$ generates $M_i\otimes N_j$.

To conclude the proof note that the space of invariants is the direct sum of trivial, one-dimensional sub representations. □

3.6. **The Brauer–Hasse–Noether theorem.** Let k be a field of characteristic $\neq 2$, let A be a central simple k-algebra (i.e. a finite-dimensional k-algebra with center k which has no non-trivial two-sided ideals). By Wedderburn's theorem, there exists a central division algebra D over k and an integer $n>0$ such that $A\simeq \mathrm{Mat}_n(D)$. Let d^2 be the dimension of D over k (this is a square because after base change, D becomes isomorphic to a matrix algebra). Then d is the *index of* A, denoted by $i(A)$. The class of A in the Brauer group of k has finite order. This integer is called the *exponent of* A, it is denoted by $e(A)$. In general, we have $e(A)|i(A)$.

Let K/k be a cyclic extension of degree n, let σ be a generator of the Galois group $\mathrm{Gal}(K/k)$, let $a\in k^*$. There is a central simple k-algebra $(\sigma,a,K/k)$ which as a k-algebra is generated by K and an element $y\in(\sigma,a,K/k)$ such that

$$y^n = a \ \text{ and } \ r\cdot y = y\cdot\sigma(r) \text{ for } r\in K.$$

This algebra is called the *cyclic algebra associated with* σ, a *and* K/k. A cyclic algebra over k of dimension 4 is a quaternion algebra.

Theorem 3.6.1 (Brauer, Hasse, Noether [BHN])**.** *Let k be an algebraic number field. Then any central division algebra A over k is a cyclic algebra (for an appropriate cyclic extension K/k and σ and a as above). Moreover, the exponent and the index of A coincide. In particular, a central division algebra of exponent 2 is a quaternion algebra.*

3.7. **An example.** We continue to assume that (T, h, q) is a Hodge structure of K3 type with $E = \mathrm{End}_{\mathrm{Hdg}}(T)$ a totally real number field of degree d over $\mathbb{Q}$. By [vG4, Prop. 3.2] we have $\dim_E T \geq 3$. We will consider now the case that $\dim_E T = 3$.

Then T_1 is an 3-dimensional $\widetilde{E}$-vector space with quadratic form q_1 of signature $(2+, 1-)$. The 3-dimensional quadratic spaces $(T_2, q_2), \ldots, (T_d, q_d)$ are negative definite. This implies that

$$C^0(q_1)_{\mathbb{R}} = \mathrm{Mat}_2(\mathbb{R}) \text{ and}$$
$$C^0(q_i)_{\mathbb{R}} = \mathbb{H} \text{ for } i \geq 2$$

(see [vG3, Thm. 7.7]). Since

$$\mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q)) \otimes_{\mathbb{Q}} \widetilde{E} = Z_G(C^0(Q)) = C^0(q_1) \otimes_{\widetilde{E}} \ldots \otimes_{\widetilde{E}} C^0(q_d)$$

we get

$$\mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q)) \otimes_{\mathbb{Q}} \mathbb{R} = \mathrm{Mat}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} \mathbb{H}.$$

Now, since $\mathbb{H} \otimes \mathbb{H} \simeq \mathrm{Mat}_4(\mathbb{R})$ this becomes

$$\mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q)) \otimes_{\mathbb{Q}} \mathbb{R} \simeq \begin{cases} \mathrm{Mat}_{2^{d-1}}(\mathbb{H}) \text{ for even } d \\ \mathrm{Mat}_{2^d}(\mathbb{R}) \text{ for odd } d. \end{cases} \tag{21}$$

On the other hand, the corestriction induces a homomorphism of Brauer groups

$$\mathrm{cores} : \mathrm{Br}(E) \to \mathrm{Br}(\mathbb{Q})$$

(cf. [D, §9, Thm. 5]). Therefore, the exponent of $\mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))$ in the Brauer group of $\mathbb{Q}$ is 2. By the Brauer–Hasse–Noether Theorem 3.6.1 there exists a (possibly split) quaternion algebra D over $\mathbb{Q}$ with

$$\mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q)) \simeq \mathrm{Mat}_{2^{d-1}}(D). \tag{22}$$

Combining (21) with (22) we see that D is a definite quaternion algebra over $\mathbb{Q}$ in case d is even and an indefinite quaternion algebra in case d is odd. The endomorphism algebra of a Kuga–Satake variety of (T, h, q) is $\mathrm{Mat}_{2^{2d-2}}(D)$. Since the dimension of a Kuga–Satake variety is $2^{\dim_{\mathbb{Q}}(T)-2} = 2^{3d-2}$, we have proved

Corollary 3.7.1. *Let (T, q, h) be a Hodge structure of K3 type with $E = \mathrm{End}_{\mathrm{Hdg}}(T)$ a totally real number field of degree d over $\mathbb{Q}$. Assume that $\dim_E(T) = 3$. Then for any Kuga–Satake variety A of (T, h, q) there exists an isogeny*

$$A \sim B^{2^{2d-2}}$$

where B is a 2^d-dimensional Abelian variety.

If d is even, B is a simple Abelian variety of type III, i.e. $\mathrm{End}_{\mathbb{Q}}(B) = D$ for a definite quaternion algebra D over $\mathbb{Q}$.

If d is odd, B has endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(B) = D$ *for an indefinite (possibly split) quaternion algebra D over $\mathbb{Q}$.*

Remark. (i) In the case $d = 2$ and $\dim_E(T) = 3$, van Geemen showed in [vG4, Prop. 5.7] that the Kuga–Satake variety of T is isogenous to a self-product of an Abelian fourfold with definite quaternion multiplication and Picard number 1. It is this case which will be of interest in the next section.

(ii) The case $d = \dim_E(T) = 3$ was also treated by van Geemen (see [vG4, 5.8 and 6.4]). He considers the case $D \simeq \mathrm{Mat}_2(\mathbb{Q})$ and relates this to work of Mumford and Galluzzi. Note that in this case the Abelian variety B of the corollary is not simple.

Example. In [vG4, 3.4], van Geemen constructs a one-dimensional family of six-dimensional K3 type Hodge structures with real multiplication by a quadratic field $E = \mathbb{Q}(\sqrt{d})$ for some square-free integer $d > 0$ which can be written in the form $d = c^2 + e^2$ for rational $c, e > 0$. These Hodge structures are realized as the transcendental lattice of certain K3 surfaces which are double covers of $\mathbb{P}^2$, see Section 4. Pick a member S of this family. Then $T(S) \otimes_{\mathbb{Q}} E$ splits in the direct sum of two three-dimensional E-vector spaces T_1 and T_2. It turns out that the quadratic space $(T_1, q_1) = (T_1, Q)$ is isometric to $(E^3, \sqrt{d}X_1^2 + \sqrt{d}X_2^2 - (d - \sqrt{d}c)X_3^2)$. Consequently

$$C^0(Q) = (-d, \sqrt{d}(d - \sqrt{d}c))_E \simeq (-1, \sqrt{d} - c)_E.$$

Here for $a, b \in E^*$, the symbol $(a, b)_E$ denotes the quaternion algebra over E generated by elements $1, i$ and j subject to the relations $i^2 = a, j^2 = b$ and $ij = -ji$ (see [vG3, Ex. 7.5]).

The projection formula for central simple algebras (see [T, Thm. 3.2]) implies that

$$\begin{aligned}\mathrm{Cores}_{E/\mathbb{Q}}(C^0(\mathbb{Q})) &\simeq (-1, N_{E/\mathbb{Q}}(\sqrt{d} - c))_{\mathbb{Q}} \\ &\simeq (-1, c^2 - d)_{\mathbb{Q}} \simeq (-1, -e^2)_{\mathbb{Q}} \simeq (-1, -1)_{\mathbb{Q}}\end{aligned}$$

which are simply Hamilton's quaternions over $\mathbb{Q}$. Here, $N_{E/\mathbb{Q}} : E \to \mathbb{Q}$ is the norm map. Hence, a Kuga–Satake variety for $T(S)$ is isogenous to a self-product B^4 where B is a simple Abelian fourfold with $\mathrm{End}_{\mathbb{Q}}(B) = (-1, -1)_{\mathbb{Q}}$.

4. Double covers of $\mathbb{P}^2$ branched along six lines

Let S be a K3 surface which admits a morphism $p : S \to \mathbb{P}^2$ such that the branch locus of p is the union of six lines.

In this section we use the decomposition theorem to prove Theorem 2 which states that the Hodge conjecture holds for $S \times S$.

4.1. **Abelian varieties of Weil type.** By a result of Lombardo [Lo], the Kuga–Satake variety of S is of Weil type. We briefly recall what this means.

Let $K = \mathbb{Q}(\sqrt{-d})$ for some square-free $d \in \mathbb{N}$. A polarized Abelian variety (A, H) of dimension $2n$ is said to be of *Weil type for K* if there is an inclusion $K \subset \mathrm{End}_{\mathbb{Q}}(A)$ mapping $\sqrt{-d}$ to φ such that

• the restriction of $\varphi^* : H^1(A,\mathbb{C}) \to H^1(A,\mathbb{C})$ to $H^{1,0}(A)$ is diagonalizable with eigenvalues $\sqrt{-d}$ and $-\sqrt{-d}$, both appearing with multiplicity n,

• $\varphi^* H = dH$.

There is a natural K-valued Hermitian form on the K-vector space $H^1(A,\mathbb{Q})$ which is defined by

$$\widetilde{H} : H^1(A,\mathbb{Q}) \times H^1(A,\mathbb{Q}) \to K, \quad (v,w) \mapsto H(\varphi^* v, w) + \sqrt{-d} H(v,w).$$

By definition, the discriminant of a polarized Abelian variety of Weil type (A,H,K) is

$$\mathrm{disc}(A,H,K) = \mathrm{disc}(\widetilde{H}) \in \mathbb{Q}^*/N_{K/\mathbb{Q}}(K^*)$$

where $N_{K/\mathbb{Q}} : K \to \mathbb{Q}$ is the norm map.

Polarized Abelian varieties of Weil type come in n^2-dimensional families (see [vG2, 5.3]).

Weil introduced such varieties as examples of Abelian varieties which carry interesting Hodge classes. He constructs a two-dimensional space, called the space of *Weil cycles*

$$W_K \subset H^{n,n}(A,\mathbb{Q}).$$

For the definition of W_K see [vG2, 5.2]. In general, the algebraicity of the classes in W_K is not known. Nonetheless there are some positive results. Here we mention one which we will use below.

Theorem 4.1.1 (Schoen [S] and van Geemen [vG1], Thm. 3.7)**.** *Let (A,H) be a polarized Abelian fourfold of Weil type for the field $\mathbb{Q}(i)$. Assume that the discriminant of $(A,H,\mathbb{Q}(i))$ is* 1*. Then the space of Weil cycles $W_{\mathbb{Q}(i)}$ is spanned by classes of algebraic cycles.*

Van Geemen uses a six-dimensional eigenspace in the complete linear system of the unique totally symmetric line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = H$ to get a rational (2:1) map of A onto a quadric $Q \subset \mathbb{P}^5$. Then the projection on $W_{\mathbb{Q}(i)}$ of the classes of the pullbacks of the two rulings of Q generate the space $W_{\mathbb{Q}(i)}$.

4.2. **Abelian varieties with quaternion multiplication.** Let D be a definite quaternion algebra over $\mathbb{Q}$. Such a D admits an involution $x \mapsto \overline{x}$ which after tensoring with $\mathbb{R}$ becomes the natural involution on Hamilton's quaternions $\mathbb{H}$.

A polarized Abelian variety (A,H) of dimension $2n$ has *quaternion multiplication by D* if there is an inclusion $D \subset \mathrm{End}_{\mathbb{Q}}(A)$ such that

• $H^1(A,\mathbb{Q})$ becomes a D-vector space and

• for $x \in D$ we have $x^* H = x\overline{x} H$.

We say that (A,H,D) is an Abelian variety of definite quaternion type. Polarized Abelian varieties of dimension $2n$ with quaternion multiplication by the same quaternion algebra come in $n(n-1)/2$-dimensional families (cf. [BL, Sect. 9.5]).

Let $K \subset D$ be a quadratic extension field of $\mathbb{Q}$. Then K is a CM field and (A, H, K) is a polarized Abelian variety of Weil type (see [vGV, Lemma 4.5]). The space of quaternion Weil cycles of (A, H, D)

$$W_D \subset H^{n,n}(A, \mathbb{Q})$$

is defined as the span of $x^* W_K$ where x runs over D. It can be shown that this is independent of the choice of K (see [vGV, Prop. 4.7]). For the general member of the family of polarized Abelian varieties with quaternion multiplication these are essentially all Hodge classes:

Theorem 4.2.1 (Abdulali, see [A], Thm. 4.1)**.** *Let (A, H, D) be a general Abelian variety of quaternion type. Then the space of Hodge classes on any self-product of A is generated by products of divisor classes and quaternion Weil cycles on A.*

In particular, if for one quadratic extension field $K \subset D$ the space of Weil cycles W_K is known to be algebraic, then the Hodge conjecture holds for any self-product of A.

In Abdulali's theorem, a triple (A, H, D) is general if the special Mumford–Tate group of $H^1(A, \mathbb{Q})$ is the maximal one. In the moduli space of triples (A, H, D) the locus of general triples is everything but a countable union of proper, closed subsets.

4.3. **The transcendental lattice of S.** We now turn back to our K3 surface S. Let $p : S \to \mathbb{P}^2$ be the (2:1) morphism which is ramified over six lines.

The Néron–Severi group of S contains the 15 classes $e_1, \dots, e_{15}$ corresponding to the exceptional divisors over the intersection points of the six lines. Let h be the class of the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$.

Define $\widetilde{T}(S) := \langle e_1, \dots, e_{15}, h \rangle^{\perp} \subset H^2(S, \mathbb{Q})$. The (rational) transcendental lattice of S is defined to be $T(S) := \mathrm{NS}(S)^{\perp} \subset H^2(S, \mathbb{Q})$. Then we have

$$T(S) \subset \widetilde{T}(S).$$

Both, $T(S)$ and $\widetilde{T}(S)$ are Hodge structures of K3 type. In addition, $T(S)$ is irreducible. Since the second Betti number of S is 22, the $\mathbb{Q}$-dimension of $\widetilde{T}(S)$ is 6.

4.4. **The Kuga–Satake variety of $\widetilde{T}(S)$.** Denote by A the Kuga–Satake variety associated with $\widetilde{T}(S)$.

Theorem 4.4.1 (Lombardo, see [Lo], Cor. 6.3 and Thm. 6.4)**.** *There is an isogeny*

$$A \sim B^4$$

where B is an Abelian fourfold with $\mathbb{Q}(i) \subset \mathrm{End}_{\mathbb{Q}}(B)$. Moreover, B admits a polarization H such that $(B, H, \mathbb{Q}(i))$ is a polarized Abelian variety of Weil type with $\mathrm{disc}(B, H, \mathbb{Q}(i)) = 1$.

Paranjape [P] explains in a very nice way how this variety B is geometrically related to S. He shows that there exists a triple

$$(C, E, f : C \to E)$$

where C is a genus five curve, E an elliptic curve and f a $(4:1)$ map such that

$$\mathrm{Prym}(f) = B.$$

Then S can be obtained as the resolution of a certain quotient of $C \times C$. It is noteworthy that Paranjape does not construct explicitly a triple (C, E, f) starting with a K3 surface S in the family π. His proof goes the other way round. He associates to any triple a K3 surface and shows then that letting vary the triple he obtains all surfaces in the family π.

Paranjape's construction establishes that the Kuga–Satake inclusion

$$\widetilde{T}(S) \hookrightarrow H^2(B^4 \times B^4, \mathbb{Q}) \tag{23}$$

is given by an algebraic cycle on $S \times B^4 \times B^4$.

4.5. **Proof of Theorem 2.** As pointed out in the introduction, we have to prove that $E_S := \mathrm{End}_{\mathrm{Hdg}}(T(S))$ is spanned by algebraic classes. Since the Picard number of S is at least 16, we can apply Ramón-Marí's corollary [RM] of Mukai's theorem [Mu1] which proves the assertion in the case that S has complex multiplication.

Therefore, we may assume that S has real multiplication. Note that $T(S)$ is an E_S-vector space and that $\dim_{E_S} T(S) \cdot [E_S : \mathbb{Q}] = \dim_{\mathbb{Q}} T(S) \leq 6$. On the other hand, by [vG4, Lemma 3.2], we know that $\dim_{E_S} T(S) \geq 3$. It follows that either $E_S = \mathbb{Q}$ or $E_S = \mathbb{Q}(\sqrt{d})$ for some square-free $d \in \mathbb{Q}_{>0}$. In the first case we use the fact, that the class of the diagonal $\Delta \subset S \times S$ induces the identity on the cohomology and that the Künneth projectors are algebraic on surfaces so that $\mathbb{Q}\,\mathrm{id} \subset E_S$ is spanned by an algebraic class.

It remains to study the case $E_S = \mathbb{Q}(\sqrt{d})$. The idea is to consider the Kuga–Satake variety $A(S)$ of $\widetilde{T}(S) = T(S)$. By Paranjape's theorem the inclusion

$$\widetilde{T}(S) \subset H^2(A(S) \times A(S), \mathbb{Q})$$

is algebraic. It follows that there is an algebraic projection $\pi : H^2(A(S) \times A(S), \mathbb{Q}) \to \widetilde{T}(S)$ (see [K, Cor. 3.14]) and therefore it is enough to show that there is an algebraic class

$$\alpha \in H^2(A(S) \times A(S), \mathbb{Q}) \otimes H^2(A(S) \times A(S), \mathbb{Q}) \subset H^4(A(S)^4, \mathbb{Q})$$

with $\pi \otimes \pi(\alpha) = \sqrt{d}$.

Combining Corollary 3.7.1 with Lombardo's theorem 4.4.1 we see that $A(S) \sim B^4$ where B is an Abelian fourfold with $\mathrm{End}_{\mathbb{Q}}(B) = D$ for a definite quaternion algebra and $\mathbb{Q}(i) \subset D$. Moreover, there is a polarization H of B such that $(B, H, \mathbb{Q}(i))$ is a polarized Abelian variety of Weil type of discriminant 1. Since by [BL, Prop. 5.5.7], the Picard number of B is 1, (B, H, D) is a polarized Abelian variety of quaternion type.

There is a one-dimensional family $(B, H, D)_t$ of deformations of (B, H, D) and this corresponds to a one-dimensional family S_t of deformations of S which parametrizes K3 surfaces with real multiplication by the same class. By Abdulali's Theorem 4.2.1, for t general the space of Hodge classes on $(B_t)^{16} \sim A(S_t)^4$ is generated by products of divisors and quaternion Weil cycles, that is by products of H and classes in W_D. Denote the span of these products in $H^4(A(S_t)^4, \mathbb{Q})$ by F_t.

Since the class corresponding to $\sqrt{d} \in \widetilde{T}(S_t) \otimes \widetilde{T}(S_t)$, the projection $\pi : H^2(A(S_t)^2, \mathbb{Q}) \rightarrow \widetilde{T}(S_t)$ and the space F_t are locally constant, there exists a locally constant class $\alpha_t \in H^4(A_{S_t}, \mathbb{Q})$ with the properties:

- for all t we have $\pi \otimes \pi(\alpha_t) = \sqrt{d}$,
- for all t we have $\alpha_t \in F_t$.

Now by Schoen's and van Geemen's Theorem 4.1.1 the space of Weil cycles $W_{\mathbb{Q}(i)}$ is generated by algebraic classes on any B_t. It follows that W_D is generated by algebraic classes and consequently F_t is generated by algebraic classes for any t. In particular, $\alpha_t \in F_t$ is algebraic. This proves the theorem. □

References

[A] S. Abdulali, *Abelian varieties of type III and the Hodge conjecture*, Int. J. Math. **10**, no. 6, (1999), 667-675.

[BL] C. Birkenhake, H. Lange, *Complex abelian varieties. Second edition*, Grundlehren, Band 302, Springer (2004).

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Erg. Math. 3. Folge, Band 21, Springer (1990).

[BHN] R. Brauer, H. Hasse, E. Noether, *Beweis eines Hauptsatzes in der Theorie der Algebren*, J. reine angew. Math. **167** (1932), 399-404.

[D] P. Draxl, *Skew fields*, LMS Lect. Note Series 31, Cambridge University Press (1983).

[vG1] B. van Geemen, *Theta functions and cycles on some abelian fourfolds*, Math. Zeit. **221** (1996), 617-631.

[vG2] B. van Geemen, *An introduction to the Hodge conjecture for Abelian varieties*, in: Algebraic Cycles and Hodge Theory (Torino, 1993), LNM 1594, Springer (1994), 233-252.

[vG3] B. van Geemen, *Kuga–Satake varieties and the Hodge conjecture*, in: The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), Kluwer, Dordrecht (2000), 51-82.

[vG4] B. van Geemen, *Real multiplication on K3 surfaces and Kuga–Satake varieties*, Mich. Math. J. **56** (2008), 375-399.

[vGV] B. van Geemen, A. Verra, *Quaternionic Pryms and Hodge classes*, Topology **42** (2003), 35-53.

[G] B. Gordon, *A Survey of the Hodge Conjecture for Abelian Varieties*, Appendix B in: J. Lewis, A Survey of the Hodge Conjecture, CRM Monograph Series 10, 2nd ed. (1999).

[GHJ] M. Gross, D. Huybrechts, D. Joyce, *Calabi–Yau manifolds and related geometries*, Springer Universitext (2002).

[K] S. Kleiman, *Algebraic cycles and the Weil conjectures*, in: Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam (1968), 359-386.

[KS] M. Kuga, I. Satake, *Abelian varieties attached to polarized K3 surfaces*, Math. Ann. **169** (1967), 239-242.

[La] S. Lang, *Algebra*, 3rd ed., Addison-Wesley (1993).

[Lo] G. Lombardo, *Abelian varieties of Weil type and Kuga–Satake varieties*, Tôhoku Math. J. **53** (2001), 453-466.

[MSY] K. Matsumoto, T. Sasaki, M. Yoshida, *The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type (3,6)*, Int. J. Math. **3** (1992), 1-164.

[Mu1] S. Mukai, *On the moduli space of bundles on K3 surfaces, I*, in: Vector Bundles on Algebraic Varieties, Bombay (1984), 341-413.

[Mu2] S. Mukai, *Vector bundles on a K3 surface*, Proc. of the ICM, Vol. II, (Beijing, 2002), 495-502.

[N] V. Nikulin, *On correspondences between surfaces of K3 type*, (in Russian), translation in: Math. USSR-Izv. **30** (1988), 375-383.

[P] K. Paranjape, *Abelian varieties associated to certain K3 surfaces*, Comp. Math. **68** (1988), 11-22.

[RM] J. Ramón-Marí, *On the Hodge conjecture for products of certain surfaces*, eprint `math.AG/055357`.

[S] C. Schoen, *Hodge classes on self-products of a variety with an automorphism*, Comp. Math. **65** (1988), 3-32.
[T] J.-P. Tignol, *On the Corestriction of Central Simple Algebras*, Math. Zeit. **194** (1987), 267-274.
[Z] Y. Zarhin, *Hodge groups of K3 surfaces*, J. reine angew. Math. **341** (1983), 193-220.

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