Dense Continuous-Time Tracking and Mapping with Rolling Shutter RGB-D Cameras Supplementary Material

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A. Derivative of Pose w.r.t. Control Points

The Jacobians defined in (14) and (15) require the derivative of a pose $\boldsymbol{T}(t) \in \mathrm{SE}(3)$ at time t w.r.t. to increments to the four control points defining the pose, *i.e.*, $\left.\frac{\partial \boldsymbol{T}(t)}{\partial \Delta \boldsymbol{T}_C}\right|_{\Delta \boldsymbol{T}_C=\mathbf{0}}$. Let $t \in [t_1, t_2)$ then $\boldsymbol{T}(t)$ is influenced by $\boldsymbol{T}_{C,0}$, $\boldsymbol{T}_{C,1}$, $\boldsymbol{T}_{C,2}$, $\boldsymbol{T}_{C,3}$. The increments are represented as 6-vectors belonging to the Lie algebra $\mathfrak{se}(3)$. Therefore, the Jacobian is a 12×24 matrix. The pose of a control point $\boldsymbol{T}_{C,l}$ is updated with an increment $\Delta \boldsymbol{T}_{C,l} \in \mathfrak{se}(3)$ with $\boldsymbol{T}_{C,l} \leftarrow \exp(\Delta \boldsymbol{T}_{C,l})\boldsymbol{T}_{C,l}$.

Writing $\boldsymbol{T}(t)$ in terms of (4) using the control points and their increments we obtain:

$$\boldsymbol{T}(t) = \exp(\Delta \boldsymbol{T}_{C,0})\boldsymbol{T}_{C,0}\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3 \tag{A.1}$$

with

$$\boldsymbol{A}_i = \exp\left(\boldsymbol{B}_i(u)\log\left(\left(\exp(\Delta \boldsymbol{T}_{C,i-1})\boldsymbol{T}_{C,i-1}\right)^{-1}\exp(\Delta \boldsymbol{T}_{C,i})\boldsymbol{T}_{C,i}\right)\right). \tag{A.2}$$

Therefore, each of the increments to $\boldsymbol{T}_{C,0}$, $\boldsymbol{T}_{C,1}$, $\boldsymbol{T}_{C,2}$ appears in two factors and the one to $\boldsymbol{T}_{C,3}$ only in $\boldsymbol{A}_3$. Writing the Jacobians for every control point separatly and applying the product rule we get:

$$\left.\frac{\partial \boldsymbol{T}(t)}{\partial \Delta \boldsymbol{T}_{C,0}}\right|_{\Delta \boldsymbol{T}_{C,0}=\mathbf{0}} = \frac{\partial\left(\exp(\Delta \boldsymbol{T}_{C,0})\boldsymbol{T}_{C,0}\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\right)}{\partial \Delta \boldsymbol{T}_{C,0}} + \frac{\partial\left(\boldsymbol{T}_{C,0}\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\right)}{\partial \boldsymbol{A}_1}\frac{\partial \boldsymbol{A}_1}{\partial \Delta \boldsymbol{T}_{C,0}} \tag{A.3}$$

$$\left.\frac{\partial \boldsymbol{T}(t)}{\partial \Delta \boldsymbol{T}_{C,1}}\right|_{\Delta \boldsymbol{T}_{C,1}=\mathbf{0}} = \frac{\partial\left(\boldsymbol{T}_{C,0}\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\right)}{\partial \boldsymbol{A}_1}\frac{\partial \boldsymbol{A}_1}{\partial \Delta \boldsymbol{T}_{C,1}} + \frac{\partial\left(\boldsymbol{T}_{C,0}\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\right)}{\partial \boldsymbol{A}_2}\frac{\partial \boldsymbol{A}_2}{\partial \Delta \boldsymbol{T}_{C,1}} \tag{A.4}$$

$$\left.\frac{\partial \boldsymbol{T}(t)}{\partial \Delta \boldsymbol{T}_{C,2}}\right|_{\Delta \boldsymbol{T}_{C,2}=\mathbf{0}} = \frac{\partial\left(\boldsymbol{T}_{C,0}\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\right)}{\partial \boldsymbol{A}_2}\frac{\partial \boldsymbol{A}_2}{\partial \Delta \boldsymbol{T}_{C,2}} + \frac{\partial\left(\boldsymbol{T}_{C,0}\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\right)}{\partial \boldsymbol{A}_3}\frac{\partial \boldsymbol{A}_3}{\partial \Delta \boldsymbol{T}_{C,2}} \tag{A.5}$$

$$\left.\frac{\partial \boldsymbol{T}(t)}{\partial \Delta \boldsymbol{T}_{C,3}}\right|_{\Delta \boldsymbol{T}_{C,3}=\mathbf{0}} = \frac{\partial\left(\boldsymbol{T}_{C,0}\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\right)}{\partial \boldsymbol{A}_3}\frac{\partial \boldsymbol{A}_3}{\partial \Delta \boldsymbol{T}_{C,3}} \tag{A.6}$$

The derivatives for the first factor in each summand can be derived using formula (7.11) and (7.12) from [1]. The last missing expressions are $\left.\frac{\partial \boldsymbol{A}_i}{\partial \Delta \boldsymbol{T}_{C,i-1}}\right|_{\Delta \boldsymbol{T}_{C,i-1}=\mathbf{0}}$ and $\left.\frac{\partial \boldsymbol{A}_i}{\partial \Delta \boldsymbol{T}_{C,i}}\right|_{\Delta \boldsymbol{T}_{C,i}=\mathbf{0}}$. Applying the chain rule we get:

$$\left.\frac{\partial \boldsymbol{A}_i}{\partial \Delta \boldsymbol{T}_{C,i}}\right|_{\Delta \boldsymbol{T}_{C,i}=\mathbf{0}} = \left.\frac{\partial \exp(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=\boldsymbol{B}_i(u)\log(\boldsymbol{T}_{C,i-1}^{-1}\boldsymbol{T}_{C,i})} \boldsymbol{B}_i(u) \left.\frac{\partial \log(\boldsymbol{T}_{C,i-1}^{-1}\boldsymbol{D}\boldsymbol{T}_{C,i})}{\partial \boldsymbol{D}}\right|_{\boldsymbol{D}=\exp(\Delta \boldsymbol{T}_{C,i})} \left.\frac{\partial \exp(\Delta \boldsymbol{T}_{C,i})}{\partial \Delta \boldsymbol{T}_{C,i}}\right|_{\Delta \boldsymbol{T}_{C,i}=\mathbf{0}} \tag{A.7}$$

For $\frac{\partial \boldsymbol{A}_i}{\partial \Delta \boldsymbol{T}_{C,i-1}}$ the expression $\log(\boldsymbol{T}_{C,i-1}^{-1} \exp(\Delta \boldsymbol{T}_{C,i}) \boldsymbol{T}_{C,i})$ changes to $\log(\boldsymbol{T}_{C,i-1}^{-1} \exp(-\Delta \boldsymbol{T}_{C,i-1}) \boldsymbol{T}_{C,i})$, let $\boldsymbol{D} = \exp(-\Delta \boldsymbol{T}_{C,i-1})$, we see that only the last factor in (A.7) is different for $\Delta \boldsymbol{T}_{C,i-1}$ and $\Delta \boldsymbol{T}_{C,i}$. It turns out that

$$\left.\frac{\partial \exp(\Delta \boldsymbol{T}_{C,i})}{\partial \Delta \boldsymbol{T}_{C,i}}\right|_{\Delta \boldsymbol{T}_{C,i}=\boldsymbol{0}} = -\left.\frac{\partial \exp(-\Delta \boldsymbol{T}_{C,i-1})}{\partial \Delta \boldsymbol{T}_{C,i-1}}\right|_{\Delta \boldsymbol{T}_{C,i-1}=\boldsymbol{0}}. \tag{A.8}$$

Therefore, we get $\left.\frac{\partial \boldsymbol{A}_i}{\partial \Delta \boldsymbol{T}_{C,i-1}}\right|_{\Delta \boldsymbol{T}_{C,i-1}=\boldsymbol{0}} = -\left.\frac{\partial \boldsymbol{A}_i}{\partial \Delta \boldsymbol{T}_{C,i}}\right|_{\Delta \boldsymbol{T}_{C,i}=\boldsymbol{0}}$. This simplifies the derivatives in (A.3), (A.4), (A.5) and (A.6), because the last summand in (A.3) is given by the negative of the first summand in (A.4) and so on. Note that (A.7) requires the Jacobian of the matrix exponential at a point different from identity. We derived an analytic expression for this Jacobian from the closed form solution of the matrix exponential using a computer algebra system. We verified that the derivative of the rotational part gives the same results as the formula derived by Gallego *et al*. [2]. It is important to have an analytic expression for this Jacobian, because it has to be evaluated for every row in the image.

As we show above, it holds that $\left.\frac{\partial \log(\boldsymbol{T}_{C,i-1}^{-1} \exp(\Delta \boldsymbol{T}_{C,i}) \boldsymbol{T}_{C,i})}{\partial \Delta \boldsymbol{T}_{C,i}}\right|_{\Delta \boldsymbol{T}_{C,i}=\boldsymbol{0}} = -\left.\frac{\partial \log(\boldsymbol{T}_{C,i-1}^{-1} \exp(-\Delta \boldsymbol{T}_{C,i-1}) \boldsymbol{T}_{C,i})}{\partial \Delta \boldsymbol{T}_{C,i-1}}\right|_{\Delta \boldsymbol{T}_{C,i-1}=\boldsymbol{0}}$. Therefore, we only need one 6×6 Jacobian matrix for every knot interval which is independent of t and which we precompute once per iteration using numerical differentiation.

References

[1] J.-L. Blanco. A tutorial on se(3) transformation parameterizations and on-manifold optimization. Technical report, University of Malaga, Sept. 2010. 1

[2] G. Gallego and A. Yezzi. A compact formula for the derivative of a 3-d rotation in exponential coordinates. *Journal of Mathematical Imaging and Vision*, 51(3):378–384, 2015. 2