Suggested Homework

Nonlinear Multiscale Methods for Image and Signal Analysis

Exercise 1. Consider the gradient flow
\[ \partial_t u(t) = -p(t), \quad p(t) \in \partial J(u(t)), u(0) = f, \]
for 1-homogeneous $J$. Show (formally), that the definition of the spectral response as
\[ S(t) = t \sqrt{\partial_t J(u(t))} \]
admits a Paseval-type identity in the sense that
\[ \int_0^\infty (S(t))^2 \, dt = \|f\|_2^2. \]
Hint: Integration by parts!

First of all note that the above exercise was missing that $\text{proj}_{\text{kern}(J)}(f) = 0$, otherwise one finds this additional quantity in the computation below. Furthermore, we need to assume finite time extinction (or at least a sufficiently rapid decay for the terms resulting from the integration by parts to vanish).

We find that
\[
\int_0^\infty (S(t))^2 \, dt = \int_0^\infty t^2 \partial_t J(u(t)) \, dt
\]
\[
= \left[ t^2 \partial_t J(u(t)) \right]_0^\infty - 2 \int_0^\infty t \partial_t J(u(t)) \, dt
\]
\[
= -2 \left[ 2tJ(u(t)) \right]_0^\infty + 2 \int_0^\infty J(u(t)) \, dt
\]
\[
= 2 \int_0^\infty \langle p(t), u(t) \rangle \, dt
\]
\[
= -2 \int_0^\infty \langle \partial_t u(t), u(t) \rangle \, dt
\]
\[
= -\int_0^\infty \partial_t \|u(t)\|_2^2 \, dt
\]
\[
= \|u(0)\|_2^2 - \lim_{t \to \infty} \|u(t)\|_2^2
\]
\[
= \|f\|_2^2
\]